The Capelli problem for $\mathfrak{gl}(m|n)$ and the spectrum of invariant differential operators

Siddhartha Sahi*, Hadi Salmasian†

August 11, 2016

Abstract

The "Capelli problem" for the symmetric pairs $(\mathfrak{gl}\times\mathfrak{gl},\mathfrak{gl})$ $(\mathfrak{gl},\mathfrak{o})$, and $(\mathfrak{gl},\mathfrak{sp})$ is closely related to the theory of Jack polynomials and shifted Jack polynomials for special values of the parameter (see [12], [15], [14], [18]). In this paper, we extend this connection to the Lie superalgebra setting, namely to the supersymmetric pairs $(\mathfrak{g},\mathfrak{k}):=(\mathfrak{gl}(m|2n),\mathfrak{osp}(m|2n))$ and $(\mathfrak{gl}(m|n)\times\mathfrak{gl}(m|n),\mathfrak{gl}(m|n))$, acting on $W:=\mathcal{S}^2(\mathbb{C}^{m|2n})$ and $\mathbb{C}^{m|n}\otimes(\mathbb{C}^{m|n})^*$.

To achieve this goal, we first prove that the center of the universal enveloping algebra of the Lie superalgebra $\mathfrak{g}$ maps surjectively onto the algebra $\mathscr{PD}(W)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant differential operators on the superspace W, thereby providing an affirmative answer to the "abstract" Capelli problem for W. Our proof works more generally for $\mathfrak{gl}(m|n)$ acting on $\mathcal{S}^2(\mathbb{C}^{m|n})$ and is new even for the "ordinary" cases ($m=0$ or $n=0$) considered by Howe and Umeda in [9].

We next describe a natural basis $\{D_\lambda\}$ of $\mathscr{PD}(W)^{\mathfrak{g}}$, that we call the Capelli basis. Using the above result on the abstract Capelli problem, we generalize the work of Kostant and Sahi [12], [15], [20] by showing that the spectrum of D_λ is given by a polynomial c_λ, which is characterized uniquely by certain vanishing and symmetry properties.

We further show that the top homogeneous parts of the eigenvalue polynomials c_λ coincide with the spherical polynomials d_λ, which arise as radial parts of $\mathfrak{k}$-spherical vectors of finite dimensional $\mathfrak{g}$-modules, and which are super-analogues of Jack polynomials. This generalizes results of Knop and Sahi [14].

Finally, we make a precise connection between the polynomials c_λ and the shifted super Jack polynomials of Sergeev and Veselov [25] for special values of the parameter. We show that the two families are related by a change of coordinates that we call the "Frobenius transform".

Keywords: Lie superalgebras, the Capelli problem, super Jack polynomials, shifted super Jack polynomials.

MSC2010: 17B10, 05E05.

1 Introduction

One of the most celebrated results in classical invariant theory is the Capelli identity for $n\times n$ matrices. It plays a fundamental role in Hermann Weyl's book [26]. The Capelli identity is profoundly connected to the representation theory of the Lie algebra $\mathfrak{gl}(n)$. The well-known article of Roger Howe [8] elucidated this connection by giving a conceptual proof of the Capelli identity.

*Department of Mathematics, Rutgers University, sahi@math.rutgers.edu.
†Department of Mathematics and Statistics, University of Ottawa, hsalmasi@uottawa.ca.

Later on, Howe and Umeda [9] generalized the Capelli identity to the setting of multiplicity-free spaces by posing and solving two general questions, which they called the *abstract* and *concrete* Capelli problems. Let $\mathfrak{g}$ be a complex reductive Lie algebra, and W be a multiplicity-free $\mathfrak{g}$-space. The abstract Capelli problem asks whether the centre $\mathbf{Z}(\mathfrak{g})$ of $\mathbf{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, maps surjectively onto the algebra $\mathscr{PD}(W)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant polynomial-coefficient differential operators on W. The concrete Capelli problem asks for explicit elements of $\mathbf{Z}(\mathfrak{g})$ whose images generate $\mathscr{PD}(W)^{\mathfrak{g}}$.

Around the same time as [9], Kostant and Sahi [12], [15] considered a slightly different question, which we shall refer to here as the Capelli *eigenvalue* problem. It turns out that the algebra $\mathscr{PD}(W)^{\mathfrak{g}}$ admits a natural basis D_λ – the *Capelli basis* – which is indexed by the monoid of highest weights of $\mathfrak{g}$-modules V_λ occurring in the symmetric algebra $\mathcal{S}(W)$. The Capelli eignevalue problem asks for determination of the eigenvalue of D_μ on V_λ. It turns out that these eignvalues are of the form $\varphi_\mu(\lambda+\rho)$, where φ_μ is a symmetric polynomial and ρ is a certain "rho-shift". Although [12], [15] are written in the context of symmetric spaces, similar ideas work for the multiplicity-free setting, see [13].

In [20], it is shown that the polynomials φ_μ are uniquely characterized by certain vanishing conditions. In fact [20] considers a general class of polynomials, depending on several parameters, and in [14] it is shown that a one-parameter subfamily of these polynomials is closely related to *Jack polynomials*. More precisely, the Knop–Sahi polynomials are inhomogeneous polynomials, but their top degree terms are the (homogenenous) Jack polynomials. In the special case where the value of the parameter θ corresponds to a symmetric space, the Knop–Sahi polynomials are the eigenvalue polynomials φ_λ, and the Jack polynomials are the spherical polynomials in V_λ. Many properties of the Knop–Sahi polynomials were subsequently proved by Okounkov and Olshanski [18], who worked with a slight modification of these polynomials which they call *shifted Jack polynomials*.

The goal of our paper is to extend this circle of ideas to the Lie superalgebra setting. The Jack polynomials for $\theta=1,\frac{1}{2},2$ are spherical polynomials of the symmetric pairs

$$(\mathfrak{gl}(n)\times\mathfrak{gl}(n),\mathfrak{gl}(n)),\ (\mathfrak{gl}(n),\mathfrak{o}(n)),\ (\mathfrak{gl}(2n),\mathfrak{sp}(2n)).$$

In the Lie superalgebra setting the last two come together and there are only two pairs to consider, namely

$$(\mathfrak{gl}(m|n)\times\mathfrak{gl}(m|n),\mathfrak{gl}(m|n)) \text{ and } (\mathfrak{gl}(m|2n),\mathfrak{osp}(m|2n)).$$

The extension of the theory to the first case is easier and, as we show in Appendix B, can be achieved by combining the work of Molev [16] with that of Sergeev–Veselov [25]. Therefore in the body of the paper we concentrate on the second case, which corresponds to the action of $\mathfrak{gl}(m|2n)$ on $\mathcal{S}^2(\mathbb{C}^{m|2n})$.

In the extension of the aforementioned theory to Lie superalgebras, at least two serious difficulties arise. The first issue is that complete reducibility of finite dimensional representations fails in the case of Lie superalgebras. Complete reducibility is crucial in the approach to the abstract Capelli problem in [9] and [6], where it is needed to split off the $\mathfrak{g}$-invariants in $\mathbf{U}(\mathfrak{g})$ and $\mathscr{PD}(W)^{\mathfrak{g}}$.

The second issue that arises is that the Cartan–Helgason Theorem is not known in full generality for Lie superalgebras. In particular, it is not immediately obvious that every $\mathfrak{gl}(m|2n)$-module that appears in $\mathscr{P}(\mathcal{S}^2(\mathbb{C}^{m|2n}))$ has an $\mathfrak{osp}(m|2n)$-invariant vector. Furthermore, in the purely even case the fact that a spherical vector is determined uniquely by its radial component follows from the KAK decomposition, which does not have an analogue in the context of supergroups. These issues complicate the formulation and proof of Theorem 3.9 and Theorem 5.8.

To achieve our goal in this paper, we have to overcome the above difficulties. Our first main result is Theorem 3.9, where we give an affirmative answer to the abstract Capelli problem of Howe and Umeda

for the Lie superalgebra $\mathfrak{gl}(m|n)$ acting on $\mathcal{S}^2(\mathbb{C}^{m|n})$. In fact in Theorem 3.9 we prove a slightly more precise statement that a $\mathfrak{gl}(m|n)$-invariant differential operator of order d is in the image of an element of $\mathbf{Z}(\mathfrak{gl}(m|n))$ which has order d with respect to the standard filtration of $\mathbf{U}(\mathfrak{gl}(m|n))$. The latter refinement is needed in the proofs of Theorem 5.8 and Theorem 6.5, which we will elaborate on below. For the superpair $(\mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n), \mathfrak{gl}(m|n))$, the corresponding abstract Capelli theorem indeed follows directly from the work of Molev [16], as explained in Appendix B (see Theorem B.1).

Our second main goal concerns the Capelli eigenvalue problem for the Lie superalgebra $\mathfrak{gl}(m|2n)$ acting on $\mathcal{S}^2(\mathbb{C}^{m|2n})$. Extending [20, Theorem 1], we show that the eigenvalues of the Capelli operators are given by polynomials c_λ given in Definition 5.2 that are characterized by suitable symmetry and vanishing conditions (see Theorem 5.9). Furthermore, in Theorem 5.8 we show that the top degree homogeneous term of the eigenvalue polynomial c_λ is equal to the spherical polynomial d_λ given in Definition 5.5. This extends the result proved by Knop and Sahi in [14] explained above. The corresponding result for the superpair $(\mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n), \mathfrak{gl}(m|n))$ is Theorem B.2. In this case the spherical and eigenvalue polynomials turn out to be the well known *supersymmetric Schur polynomials* and *shifted supersymmetric Schur* polynomials [16].

Our third main goal is to establish a precise relation between our eigenvalue polynomials c_λ and certain polynomials called *shifted super Jack polynomials.* In connection with the eigenstates of the deformed Calogero–Moser–Sutherland operators [25], Sergeev and Veselov define a family of $(m+n)$-variable polynomials $SP_\flat^*$, parametrized by (m,n)-hook partitions $\flat$ (see Definition 4.3). In the case of the superpair $(\mathfrak{gl}(m|2n), \mathfrak{osp}(m|2n))$, in Proposition 6.2 and Theorem 6.5, we prove that the c_λ and the $SP_\flat^*$ are related by the *Frobenius transform*, see Definition 6.4. The corresponding statement for the pair $(\mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n), \mathfrak{gl}(m|n))$ is Theorem B.3.

It is worth mentioning that in [25], the shifted super Jack polynomials are obtained as the image of the shifted Jack polynomials under a certain *shifted Kerov map*, and the fact that the image of the shifted Kerov map is a polynomial is indeed a nontrivial statement which is proved in [25] indirectly. However, our definition of c_λ is more conceptual, and the proofs are more straightforward, as they are based on the Harish–Chandra homomorphism and the solution of the abstract Capelli problem. Furthermore, the work of Sergeev and Veselov does not address the relation with spherical representations of Lie superalgebras. Our paper establishes this connection.

We now outline the structure of this article. Section 2 defines the basic notation that is used throughout the rest of the paper. The solution to the abstract Capelli problem for $\mathfrak{gl}(m|n)$ acting on $\mathcal{S}^2(\mathbb{C}^{m|n})$ is given in Section 3. In Section 4 we study spherical highest weight modules of $\mathfrak{gl}(m|2n)$. We prove in Proposition 4.6 and Remark 4.7 that every irreducible $\mathfrak{gl}(m|2n)$-submodule of $\mathcal{S}(\mathcal{S}^2(\mathbb{C}^{m|2n}))$ or $\mathscr{P}(\mathcal{S}^2(\mathbb{C}^{m|2n}))$ has a unique (up to scalar) nonzero $\mathfrak{osp}(m|2n)$-fixed vector. It is worth mentioning that Proposition 4.6 does not follow from the work of Alldridge and Schmittner [1], since they need to assume that the highest weight is "high enough" in some sense. In Section 5, we prove Theorem 5.8 and Theorem 5.9 (see the second goal above). Section 6 is devoted to connecting the eigenvalue polynomials c_λ to the shifted super Jack polynomials of Sergeev and Veselov [25]. Appendix A contains the proof of Proposition 5.6. Finally, in Appendix B we outline the proofs of our main theorems for the case of $\mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n)$ acting on $\mathbb{C}^{m|n} \otimes (\mathbb{C}^{m|n})^*$.

We now briefly describe the structure of our proofs. There is no mystery in the formulation of the statement of Theorem 3.9, but its proof is *not* a simple generalization of any of the existing proofs in the purely even case (i.e., when $n = 0$) that we are aware of. Our approach is inspired by the proof given by Goodman and Wallach in [6, Sec. 5.7.1], but it diverges quickly because their argument relies heavily on the complete reducibility of rational representations of a reductive algebraic group. Our proof proceeds by

induction, after we show that the symbol of an invariant differential operator is in the image of $\mathbf{Z}(\mathfrak{gl}(m|n))$. This symbol is in the span of invariant tensors $\mathbf{t}_\sigma$ defined in (21) where σ is a permutation. The next step is to reduce the latter problem to the case where σ is a cycle of consecutive letters. Finally, we prove that in this special case, $\mathbf{t}_\sigma$ is in the image of the *Gelfand elements* $\mathrm{str}(\mathbf{E}^d) \in \mathbf{Z}(\mathfrak{gl}(m|n))$ defined in Lemma 3.1.

The idea behind the proof of Theorem 5.8 is as follows. Let D_λ be a Capelli operator, and let $z_\lambda \in \mathbf{Z}(\mathfrak{gl}(m|2n))$ be an element in the inverse image of D_λ, whose existence follows from Theorem 3.9. We show that modulo the natural isomorphism induced by the trace form, the spherical polynomial d_λ is the diagonal restriction of the symbol of the polynomial-coefficient differential operator corresponding to the radial part of z_λ. Furthermore, we show that the eigenvalue polynomial c_λ is the image of the radial part of z_λ (see Lemma 5.1). We combine the latter two statements, as well as the refinement of the abstract Capelli problem obtained in Theorem 3.9, to prove Theorem 5.8.

Finally, the proof of Theorem 6.5 goes as follows. By considering the action of z_λ on the lowest weight vector of an irreducible $\mathfrak{gl}(m|2n)$-module V_μ, we prove in Proposition 6.2 that $c_\lambda(\mu)$ is a polynomial in the highest weight μ^* of the contragredient representation V_μ^*. Denoting the latter polynomial c_λ^*, we verify that after a *Frobenius transform* (see Definition 6.4), the polynomial c_λ^* satisfies the supersymmetry and vanishing properties of the shifted super Jack polynomials of [25]. It then follows that the two polynomials coincide up to a scalar multiple.

We now elaborate on some of the new techniques and ideas introduced in our paper. Our method of proof of Theorem 3.9 yields a recursive procedure for expressing a given invariant differential operator explicitly as the image of an element of $\mathbf{Z}(\mathfrak{gl}(m|n))$ using Gelfand elements. In the setting of ordinary Lie algebras, Howe and Umeda [9] obtain such a formula for the *generators* of the algebra of invariant differential operators. By contrast, even in this setting, our construction is more general and gives explicit pre-images for a *basis* of the algebra.

Another new idea is the construction of spherical vectors in tensor representations of $\mathfrak{gl}(m|2n)$ using symbols of the Capelli operators D_λ (see Lemma 5.7).

Acknowledgement. We thank the referee for useful comments and pointing to references for Lemma 3.1. Throughout this project, we benefited from discussions with Alexander Alldridge, Ivan Dimitrov, Roe Goodman, Roger Howe, Friedrich Knop, James Lepowsky, Nolan Wallach, and Tilmann Wurzbacher. We thank them for fruitful and encouraging discussions. The second author is supported by an NSERC Discovery Grant.

2 Notation

We briefly review the basic theory of vector superspaces and Lie superalgebras. For more detailed expositions, see for example [5] or [17]. Throughout this article, all vector spaces will be over $\mathbb{C}$. Let SVec be the symmetric monoidal category of $\mathbb{Z}/2$-graded vector spaces, where $\mathbb{Z}/2 := \{\overline{\mathbf{0}}, \overline{\mathbf{1}}\}$. Objects of SVec are of the form $U = U_{\overline{\mathbf{0}}} \oplus U_{\overline{\mathbf{1}}}$. The parity of a homogeneous vector $u \in U$ is denoted by $|u| \in \mathbb{Z}/2$. For any two $\mathbb{Z}/2$-graded vector spaces U and U', the vector space $\mathrm{Hom}_{\mathbb{C}}(U, U')$ is naturally $\mathbb{Z}/2$-graded, and the morphisms of SVec are defined by $\mathsf{Mor}_{\mathsf{SVec}}(U, U') := \mathrm{Hom}_{\mathbb{C}}(U, U')_{\overline{\mathbf{0}}}$. The symmetry isomorphism of SVec is defined by

$$\mathbf{s}_{U,U'} : U \otimes U' \to U' \otimes U \ , \ u \otimes u' \mapsto (-1)^{|u|\cdot|u'|} u' \otimes u. \tag{1}$$

We remark that throughout this article, the defining relations which involve parities of vectors should first be construed as relations for homogeneous vectors, and then be extended by linearity to arbitrary vectors.

The identity element of the associative superalgebra $\mathrm{End}_{\mathbb{C}}(U) := \mathrm{Hom}_{\mathbb{C}}(U, U)$ will be denoted by 1_U. Note that $\mathrm{End}_{\mathbb{C}}(U)$ is a Lie superalgebra with the standard commutator $[A, B] := AB - (-1)^{|A|\cdot|B|}BA$. Set $U^* := \mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C}^{1|0})$. The map

$$U' \otimes U^* \to \mathrm{Hom}_{\mathbb{C}}(U, U')\ ,\ u' \otimes u^* \mapsto T_{u' \otimes u^*}, \tag{2}$$

where $T_{u'\otimes u^*}(u) := \langle u^*, u\rangle u'$ for all $u \in U$, is an isomorphism in the category SVec.

The natural representation (in the symmetric monoidal category SVec) of the symmetric group S_d on $U^{\otimes d}$ is explicitly given by

$$\sigma \mapsto T^\sigma_{U,d}\ \ ,\ \ T^\sigma_{U,d}(v_1 \otimes \cdots \otimes v_d) := (-1)^{\epsilon\left(\sigma^{-1}; v_1,\ldots,v_d\right)} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)}\ , \tag{3}$$

where

$$\epsilon(\sigma; v_1, \ldots, v_d) := \sum_{\substack{1 \le r < s \le d \\ \sigma(r) > \sigma(s)}} |v_{\sigma(r)}| \cdot |v_{\sigma(s)}|. \tag{4}$$

The supersymmetrization map $\mathsf{sym}^d_U : U^{\otimes d} \to U^{\otimes d}$ is defined by

$$\mathsf{sym}^d_U := \frac{1}{d!} \sum_{\sigma \in S_d} T^\sigma_{U,d}. \tag{5}$$

We set

$$\mathcal{S}^d(U) := \mathsf{sym}^d_U \left(U^{\otimes d}\right),\ \mathcal{S}(U) := \bigoplus_{d \ge 0} \mathcal{S}^d(U),\ \mathscr{P}(U) := \mathcal{S}(U^*),\ \text{and}\ \mathscr{P}^d(U) := \mathcal{S}^d(U^*).$$

Every $\eta \in U^*_{\mathbf{0}}$ extends canonically to a homomorphism of superalgebras

$$\mathsf{h}_\eta : \mathcal{S}(U) \to \mathbb{C} \cong \mathbb{C}^{1|0} \tag{6}$$

defined by $\mathsf{h}_\eta\big(\mathsf{sym}^d_U(u_1 \otimes \cdots \otimes u_d)\big) := \eta(u_1) \cdots \eta(u_d)$ for $d \ge 0$ and $u_1, \ldots, u_d \in U$.

If $\mathfrak{g}$ is a Lie superalgebra, then $\mathbf{U}(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$, and

$$\mathbb{C} = \mathbf{U}^0(\mathfrak{g}) \subset \mathbf{U}^1(\mathfrak{g}) \subset \cdots \subset \mathbf{U}^d(\mathfrak{g}) \subset \cdots$$

denotes the standard filtration of $\mathbf{U}(\mathfrak{g})$. If (π, U) is a $\mathfrak{g}$-module, we define

$$U^{\mathfrak{g}} := \{u \in U\ \ :\ \ \pi(x)u = 0 \text{ for every } x \in \mathfrak{g}\}.$$

If (π', U') is another $\mathfrak{g}$-module, then $\mathrm{Hom}_{\mathbb{C}}(U, U')$ is also a $\mathfrak{g}$-module, with the action

$$(x \cdot T)u := \pi'(x)Tu - (-1)^{|T||x|}T\pi(x)u \tag{7}$$

for $x \in \mathfrak{g}$, $T \in \mathrm{Hom}_{\mathbb{C}}(U, U')$, and $u \in U$. The special case $U^* := \mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C}^{1|0})$, where $\mathbb{C}^{1|0}$ is the trivial $\mathfrak{g}$-module, is the *contragredient* $\mathfrak{g}$-module. Moreover, the map (2) is $\mathfrak{g}$-equivariant. Set

$$\mathrm{Hom}_{\mathfrak{g}}(U,U') := \mathrm{Hom}_{\mathbb{C}}(U,U')^{\mathfrak{g}} \quad\text{and}\quad \mathrm{End}_{\mathfrak{g}}(U) := \mathrm{End}_{\mathbb{C}}(U)^{\mathfrak{g}}.$$

The category of $\mathfrak{g}$-modules is a symmetric monoidal category, and

$$\mathsf{Mor}_{\mathfrak{g}\text{-mod}}(U,U') \cong \mathrm{Hom}_{\mathfrak{g}}(U,U')_{\overline{\mathbf{0}}}.$$

Fix a $\mathbb{Z}/2$-graded vector space W. For every homogeneous $w \in W$, let ∂_w be the superderivation of $\mathscr{P}(W)$ with parity $|w|$ that is defined uniquely by

$$\partial_w(w^*) := (-1)^{|w|}\langle w^*, w\rangle \text{ for every } w^* \in W^* \cong \mathscr{P}^1(W). \tag{8}$$

Thus, $\partial_w(a_1a_2) = (\partial_w a_1)a_2 + (-1)^{|w|\cdot|a_1|}a_1\partial_w a_2$ for every $a_1, a_2 \in \mathscr{P}(W)$. For every $b \in \mathcal{S}(W)$ we define $\partial_b \in \mathrm{End}_{\mathbb{C}}(\mathscr{P}(W))$ as follows. First we set

$$\partial_{w_1\cdots w_r} := \partial_{w_1}\cdots\partial_{w_r} \text{ for homogeneous } w_1,\dots,w_r \in W,$$

and then we extend the definition of ∂_b to all $b \in \mathcal{S}(W)$ by linearity.

Let $\mathscr{PD}(W)$ be the associative superalgebra of polynomial-coefficient differential operators on W. More explicitly, $\mathscr{PD}(W)$ is the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathscr{P}(W))$ spanned by elements of the form $a\partial_b$, where $a \in \mathscr{P}(W)$ and $b \in \mathcal{S}(W)$. If W is a $\mathfrak{g}$-module, then $\mathscr{PD}(W)$ is a $\mathfrak{g}$-invariant subspace of $\mathrm{End}_{\mathbb{C}}(\mathscr{P}(W))$ with respect to the $\mathfrak{g}$-action on $\mathrm{End}_{\mathbb{C}}(\mathscr{P}(W))$ defined in (7). Furthermore, the map

$$\mathsf{m} : \mathscr{P}(W)\otimes\mathcal{S}(W) \to \mathscr{PD}(W)\ ,\ a\otimes b \mapsto a\partial_b \tag{9}$$

is an isomorphism of $\mathfrak{g}$-modules (but not of superalgebras). The superalgebra $\mathscr{PD}(W)$ has a natural filtration given by

$$\mathscr{PD}^d(W) := \mathsf{m}\left(\mathscr{P}(W)\otimes\left(\bigoplus_{r=0}^{d}\mathcal{S}^r(W)\right)\right) \text{ for } d \geq 0.$$

For $D \in \mathscr{PD}(W)$, we write $\mathrm{ord}(D) = d$ if $D \in \mathscr{PD}^d(W)$ but $D \notin \mathscr{PD}^{d-1}(W)$. For every $d \geq 0$, the d-th order symbol map

$$\widehat{\mathsf{s}}_d : \mathscr{PD}^d(W) \to \mathscr{P}(W)\otimes\mathcal{S}^d(W)$$

is defined by

$$\widehat{\mathsf{s}}_d\left(\mathscr{PD}^{d-1}(W)\right) = 0 \text{ and } \widehat{\mathsf{s}}_d(\mathsf{m}(a)) = a \text{ for } a \in \mathscr{P}(W)\otimes\mathcal{S}^d(W). \tag{10}$$

If $D_r \in \mathscr{PD}^{d_r}(W)$ for $1 \leq r \leq k$, then

$$\widehat{\mathsf{s}}_{d_1+\cdots+d_k}(D_1\dots D_k) = \widehat{\mathsf{s}}_{d_1}(D_1)\cdots\widehat{\mathsf{s}}_{d_k}(D_k), \tag{11}$$

where the product on the right hand side takes place in the superalgebra $\mathscr{P}(W)\otimes\mathcal{S}(W)$.

3 The abstract Capelli problem for $W := \mathcal{S}^2(V)$

Fix $V := \mathbb{C}^{m|n}$ and set $\mathfrak{g} := \mathfrak{gl}(V) := \mathfrak{gl}(m|n)$. We fix bases $\mathbf{e}_1, \ldots, \mathbf{e}_m$ for $V_{\overline{\mathbf{0}}} = \mathbb{C}^{m|0}$ and $\mathbf{e}_{\overline{1}}, \ldots, \mathbf{e}_{\overline{n}}$ for $V_{\overline{\mathbf{1}}} = \mathbb{C}^{0|n}$. Throughout this article we will use the index set

$$\mathcal{I}_{m,n} := \{1, \ldots, m, \overline{1}, \ldots, \overline{n}\}.$$

We define the parities of elements of $\mathcal{I}_{m,n}$ by

$$|i| := |\mathbf{e}_i| = \begin{cases} \overline{\mathbf{0}} & \text{for } i \in \{1, \ldots, m\}, \\ \overline{\mathbf{1}} & \text{for } i \in \{\overline{1}, \ldots, \overline{n}\}. \end{cases}$$

For an associative superalgebra $\mathcal{A} = \mathcal{A}_{\overline{\mathbf{0}}} \oplus \mathcal{A}_{\overline{\mathbf{1}}}$, let $\mathrm{Mat}_{m,n}(\mathcal{A})$ be the associative superalgebra of $(m + n) \times (m + n)$ matrices with entries in $\mathcal{A}$, endowed with the $\mathbb{Z}_2$-grading obtained by the (m, n)-block form of its elements. More precisely, $\mathrm{Mat}_{m,n}(\mathcal{A})_{\overline{\mathbf{0}}}$ consists of matrices in (m, n)-block form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{12}$$

such that the entries of the $m \times m$ matrix A and the $n \times n$ matrix D belong to $\mathcal{A}_{\overline{\mathbf{0}}}$, whereas the entries of the matrices B and C belong to $\mathcal{A}_{\overline{\mathbf{1}}}$. Similarly, $\mathrm{Mat}_{m,n}(\mathcal{A})_{\overline{\mathbf{1}}}$ consists of matrices of the form (12) such that the entries of A and D belong to $\mathcal{A}_{\overline{\mathbf{1}}}$, whereas the entries of B and C belong to $\mathcal{A}_{\overline{\mathbf{0}}}$. The supertrace of an element of $\mathrm{Mat}_{m,n}(\mathcal{A})$ is defined by

$$\mathrm{str}\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \mathrm{tr}(A) - \mathrm{tr}(D).$$

Using the basis $\{\mathbf{e}_i : i \in \mathcal{I}_{m,n}\}$, we can represent elements of $\mathfrak{g}$ by $(m + n) \times (m + n)$ matrices, with rows and columns indexed by elements of $\mathcal{I}_{m,n}$. For every $i, j \in \mathcal{I}_{m,n}$, let $E_{i,j}$ denote the element of $\mathfrak{g}$ corresponding to the matrix with a 1 in the (i, j)-entry and 0's elsewhere. The standard Cartan subalgebra of $\mathfrak{g}$ is

$$\mathfrak{h} := \mathrm{Span}_{\mathbb{C}}\{E_{1,1}, \ldots, E_{m,m}, E_{\overline{1},\overline{1}}, \ldots, E_{\overline{n},\overline{n}}\},$$

and the standard characters of $\mathfrak{h}$ are $\varepsilon_1, \ldots, \varepsilon_m, \varepsilon_{\overline{1}}, \ldots, \varepsilon_{\overline{n}} \in \mathfrak{h}^*$, where $\varepsilon_i(E_{j,j}) = \delta_{i,j}$ for every $i, j \in \mathcal{I}_{m,n}$. The standard root system of $\mathfrak{g}$ is $\Phi := \Phi^+ \cup \Phi^-$ where

$$\Phi^+ := \left\{\varepsilon_k - \varepsilon_l\right\}_{1 \le k < l \le m} \cup \left\{\varepsilon_k - \varepsilon_{\overline{l}}\right\}_{1 \le k \le m,\, 1 \le l \le n} \cup \left\{\varepsilon_{\overline{k}} - \varepsilon_{\overline{l}}\right\}_{1 \le k < l \le n}$$

and $\Phi^- := -\Phi^+$. For $\alpha \in \Phi$, we set $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for every } h \in \mathfrak{h}\}$. Then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^\pm := \bigoplus_{\alpha \in \Phi^\pm} \mathfrak{g}_\alpha$. Finally, the supersymmetric bilinear form

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}\ ,\ \kappa(X, Y) := \mathrm{str}(XY) \text{ for } X, Y \in \mathfrak{g}, \tag{13}$$

is nondegenerate and $\mathfrak{g}$-invariant.

Set $W := \mathcal{S}^2(V)$. The standard representation of $\mathfrak{g}$ on V gives rise to canonically defined representations on $V^{\otimes d}$, W, W^*, $\mathcal{S}(W)$, and $\mathscr{P}(W)$. From now on, we denote the latter representations of $\mathfrak{g}$ on $\mathcal{S}(W)$ and $\mathscr{P}(W)$ by ρ and $\check{\rho}$ respectively.

The goal of the rest of this section is to prove Theorem 3.9, which gives an affirmative answer to the abstract Capelli problem for the $\mathfrak{g}$-module W. Set

$$\mathbf{Z}(\mathfrak{g}) := \{x \in \mathbf{U}(\mathfrak{g}) \,:\, [x, y] = 0 \text{ for every } y \in \mathbf{U}(\mathfrak{g})\},$$

where $[\cdot, \cdot]$ denotes the standard superbracket of $\mathbf{U}(\mathfrak{g})$.

Lemma 3.1. *Let* $\mathbf{E} \in \mathrm{Mat}_{m,n}(\mathbf{U}(\mathfrak{g}))_{\overline{\mathbf{0}}}$ *be the matrix with entries*

$$\mathbf{E}_{i,j} := (-1)^{|i|\cdot|j|} E_{j,i} \text{ for } i, j \in \mathcal{I}_{m,n}.$$

Then $\mathrm{str}(\mathbf{E}^d) \in \mathbf{Z}(\mathfrak{g})$ *for every* $d \geq 1$.

Proof. This lemma can be found for example in [22] or [21]. For the reader's convenience, we outline a proof (in the case $n = 0$ it is due to Žhelobenko [27]).

Step 1. Let $\mathbf{Z} = [z_{i,j}]_{i,j \in \mathcal{I}_{m,n}}$ be an element of $\mathrm{Mat}_{m,n}(\mathbf{U}(\mathfrak{g}))_{\overline{\mathbf{0}}}$ that satisfies

$$[z_{i,j}, E_{k,l}] = (-1)^{|i|\cdot|j|+|l|\cdot|j|}\delta_{i,k} z_{l,j} - (-1)^{|j|\cdot|k|+|j|}\delta_{l,j} z_{i,k} \text{ for } i, j, k, l \in \mathcal{I}_{m,n}. \tag{14}$$

Set $z := \mathrm{str}(\mathbf{Z}) = \sum_{i \in \mathcal{I}_{m,n}} (-1)^{|i|} z_{i,i}$. Then $[z, E_{k,l}] = 0$ for $k, l \in \mathcal{I}_{m,n}$, and thus $z \in \mathbf{Z}(\mathfrak{g})$.

Step 2. Let $\mathbf{Z} = [z_{i,j}]_{i,j \in \mathcal{I}_{m,n}}$ and $\mathbf{Z}' = [z'_{i,j}]_{i,j \in \mathcal{I}_{m,n}}$ be elements of $\mathrm{Mat}_{m,n}(\mathbf{U}(\mathfrak{g}))_{\overline{\mathbf{0}}}$ that satisfy (14). Set $\mathbf{Z}'' := \mathbf{Z}\mathbf{Z}'$, so that $\mathbf{Z}'' = [z''_{i,j}]_{i,j \in \mathcal{I}_{m,n}}$ where $z''_{i,j} := \sum_{r \in \mathcal{I}_{m,n}} z_{i,r} z'_{r,j}$. Then

$$\begin{aligned}
\left[z''_{i,j}, E_{k,l}\right] &= \sum_{r \in \mathcal{I}_{m,n}} z_{i,r}[z'_{r,j}, E_{k,l}] + \sum_{r \in \mathcal{I}_{m,n}} (-1)^{(|r|+|j|)(|k|+|l|)}[z_{i,r}, E_{k,l}] z'_{r,j} \\
&= (-1)^{|j|\cdot|i|+|j|\cdot|l|}\delta_{i,k} z''_{l,j} - (-1)^{|j|\cdot|k|+|j|}\delta_{l,j} z''_{i,k},
\end{aligned}$$

so that the entries of $\mathbf{Z}''$ satisfy (14) as well.

Step 3. Since (14) holds for the entries of $\mathbf{E}$, induction on d and Step 2 imply that (14) holds for the entries of $\mathbf{E}^d$ for every $d \geq 1$, and thus $\mathrm{str}(\mathbf{E}^d) \in \mathbf{Z}(\mathfrak{g})$ by Step 1. $\square$

Let $\{\mathbf{e}_i^* \,:\, i \in \mathcal{I}_{m,n}\}$ be the basis of V^* dual to the basis $\{\mathbf{e}_i \,:\, i \in \mathcal{I}_{m,n}\}$. Set

$$x_{i,j} := \frac{1}{2}\left(\mathbf{e}_i \otimes \mathbf{e}_j + (-1)^{|i|\cdot|j|}\mathbf{e}_j \otimes \mathbf{e}_i\right) \text{ and } y_{i,j} := \mathbf{e}_j^* \otimes \mathbf{e}_i^* + (-1)^{|i|\cdot|j|}\mathbf{e}_i^* \otimes \mathbf{e}_j^*, \tag{15}$$

for $i, j \in \mathcal{I}_{m,n}$. The $x_{i,j}$ span W, and therefore they generate $\mathcal{S}(W)$. Similarly, the $y_{i,j}$ span W^*, and therefore they generate $\mathscr{P}(W)$. Moreover,

$$x_{i,j} = (-1)^{|i|\cdot|j|} x_{j,i}, \;\; y_{i,j} = (-1)^{|i|\cdot|j|} y_{j,i}, \text{ and } \langle y_{i,j}, x_{p,q}\rangle = \delta_{i,p}\delta_{j,q} + (-1)^{|i|\cdot|j|}\delta_{i,q}\delta_{j,p}.$$

The action of $\mathfrak{g}$ on $\mathcal{S}(W)$ can be realized by the polarization operators

$$\rho(E_{i,j}) := \sum_{r \in \mathcal{I}_{m,n}} x_{i,r}\mathsf{D}_{j,r} \text{ for } i, j \in \mathcal{I}_{m,n}, \tag{16}$$

where $\mathsf{D}_{i,j} : \mathcal{S}(W) \to \mathcal{S}(W)$ is the superderivation of parity $|i| + |j|$ uniquely defined by

$$\mathsf{D}_{i,j}(x_{k,l}) := \delta_{i,k}\delta_{j,l} + (-1)^{|i|\cdot|j|}\delta_{i,l}\delta_{j,k} \text{ for } i,j,k,l \in \mathcal{I}_{m,n}. \tag{17}$$

Similarly, the action of $\mathfrak{g}$ on $\mathscr{P}(W)$ is realized by the polarization operators

$$\check{\rho}(E_{i,j}) = -(-1)^{|i|\cdot|j|} \sum_{r \in \mathcal{I}_{m,n}} (-1)^{|r|} y_{r,j}\partial_{r,i} \ \text{ for } i,j \in \mathcal{I}_{m,n}, \tag{18}$$

where $\partial_{i,j} := \partial_{x_{i,j}}$ is the superderivation of $\mathscr{P}(W)$ corresponding to $x_{i,j} \in W$, as in (8).

Let $U = U_{\overline{0}} \oplus U_{\overline{1}}$ be a $\mathbb{Z}/2$-graded vector space. For every $\mathcal{A} \subseteq \mathrm{End}_{\mathbb{C}}(U)$, we set

$$\mathcal{A}' := \{B \in \mathrm{End}_{\mathbb{C}}(U) \ : \ [A,B] = 0 \text{ for every } A \in \mathcal{A}\}.$$

Remark 3.2. Fix finite dimensional $\mathbb{Z}/2$-graded vector spaces U and U', and set

$$\mathcal{A} := \mathrm{End}_{\mathbb{C}}(U) \otimes 1_{U'} \subset \mathrm{End}_{\mathbb{C}}(U \otimes U').$$

Then $\mathcal{A}' = 1_U \otimes \mathrm{End}_{\mathbb{C}}(U')$.

Lemma 3.3. *Let $U = U_{\overline{0}} \oplus U_{\overline{1}}$ be a finite dimensional $\mathbb{Z}/2$-graded vector space, and let $\mathcal{A} \subseteq \mathrm{End}_{\mathbb{C}}(U)_{\overline{0}}$ be a semisimple associative algebra. Then $(\mathcal{A}')' = \mathcal{A}$.*

Proof. Since $\mathcal{A}$ is semisimple and purely even, both $U_{\overline{0}}$ and $U_{\overline{1}}$ can be expressed as direct sums of irreducible $\mathcal{A}$-modules. It follows that $U \cong \bigoplus_\tau U_\tau \otimes V_\tau$, where the U_τ are mutually non-isomorphic irreducible $\mathcal{A}$-modules and $V_\tau \cong \mathrm{Hom}_{\mathcal{A}}(U_\tau, U)$. Since the decomposition of U into irreducibles is homogeneous, the multiplicity spaces V_τ are $\mathbb{Z}/2$-graded, whereas the U_τ are purely even. By Artin-Wedderburn theory [10], $\mathcal{A} = \bigoplus_\tau \mathrm{End}_{\mathbb{C}}(U_\tau) \otimes 1_{V_\tau}$. Thus by Remark 3.2, $\mathcal{A}' = \bigoplus_\tau 1_{U_\tau} \otimes \mathrm{End}_{\mathbb{C}}(V_\tau)$ and $(\mathcal{A}')' = \bigoplus_\tau \mathrm{End}_{\mathbb{C}}(U_\tau) \otimes 1_{V_\tau} = \mathcal{A}$. ☐

Lemma 3.4. *For every $d \geq 1$, the superalgebra $\mathrm{End}_{\mathfrak{g}}(V^{\otimes d})$ is spanned by the operators $T^\sigma_{V,d}$ defined in (3). In particular, $\mathrm{End}_{\mathfrak{g}}(V^{\otimes d}) = \mathrm{End}_{\mathfrak{g}}(V^{\otimes d})_{\overline{0}} = \mathsf{Mor}_{\mathfrak{g}\text{-mod}}(V^{\otimes d}, V^{\otimes d})$.*

Proof. Set $\mathcal{A} := \mathrm{Span}_{\mathbb{C}}\{T^\sigma_{V,d} \ : \ \sigma \in S_d\} \subset \mathrm{End}_{\mathbb{C}}(V^{\otimes d})_{\overline{0}}$. Since $\mathcal{A}$ is a homomorphic image of the group algebra $\mathbb{C}[S_d]$, it is a semisimple associative algebra, and Lemma 3.3 implies that $(\mathcal{A}')' = \mathcal{A}$.

Let $\mathcal{B}$ be the associative subalgebra of $\mathrm{End}_{\mathbb{C}}(V^{\otimes d})$ generated by the image of $\mathbf{U}(\mathfrak{g})$. From [23, Theorem 1] or [2, Theorem 4.14] it follows that $\mathcal{A}' = \mathcal{B}$. Therefore we obtain that $\mathrm{End}_{\mathfrak{g}}(V^{\otimes d}) = \mathcal{B}' = (\mathcal{A}')' = \mathcal{A}$. ☐

The canonical isomorphism

$$V^{\otimes 2d} \otimes V^{*\otimes 2d} \cong \mathrm{End}_{\mathbb{C}}(V^{\otimes 2d}) \tag{19}$$

given in (2) maps $v_1 \otimes \cdots \otimes v_{2d} \otimes v_1^* \otimes \cdots \otimes v_{2d}^* \in V^{\otimes 2d} \otimes V^{*\otimes 2d}$ to the linear map

$$V^{\otimes 2d} \to V^{\otimes 2d} \ , \ w_1 \otimes \cdots \otimes w_{2d} \mapsto \langle v_{2d}^*, w_1\rangle \cdots \langle v_1^*, w_{2d}\rangle v_1 \otimes \cdots \otimes v_{2d}.$$

For every $\sigma \in S_{2d}$, let $\widetilde{\mathbf{t}}_\sigma \in V^{\otimes 2d} \otimes V^{*\otimes 2d}$ be the element corresponding to $T^{\sigma^{-1}}_{V,2d}$ via the isomorphism (19), where $T^{\sigma^{-1}}_{V,2d}$ is defined as in (3). It is easy to verify that

$$\widetilde{\mathbf{t}}_\sigma = \sum_{i_1,\ldots,i_d \in \mathcal{I}_{m,n}} (-1)^{\epsilon\left(\sigma; \mathbf{e}_{i_1},\ldots,\mathbf{e}_{i_{2d}}\right)} \mathbf{e}_{i_{\sigma(1)}} \otimes \cdots \otimes \mathbf{e}_{i_{\sigma(2d)}} \otimes \mathbf{e}^*_{i_{2d}} \otimes \cdots \otimes \mathbf{e}^*_{i_1},$$

where $\epsilon\left(\sigma; \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_{2d}}\right)$ is defined in (4). Now set $\sigma_\circ := (1,2)\cdots(2d-1,2d) \in S_{2d}$ and

$$H_{2d} := \{\sigma \in S_{2d} \,:\, \sigma\sigma_\circ = \sigma_\circ\sigma\}.$$

Let $\mathsf{p}_d : V^{\otimes 2d} \to V^{\otimes 2d}$ and $\mathsf{p}_d^* : V^{*\otimes 2d} \to V^{*\otimes 2d}$ be the canonical projections onto $\mathcal{S}^d(W)$ and $\mathcal{S}^d(W^*) \cong \mathscr{P}^d(W)$, so that

$$\mathsf{p}_d = \frac{1}{2^d d!}\sum_{\sigma\in H_{2d}} T^\sigma_{V,2d} \quad\text{and}\quad \mathsf{p}_d^* = \frac{1}{2^d d!}\sum_{\sigma\in H_{2d}} T^\sigma_{V^*,2d}. \tag{20}$$

For every $\sigma \in S_{2d}$, let $\mathbf{t}_\sigma \in \mathscr{P}^d(W)\otimes\mathcal{S}^d(W)$ be defined by

$$\mathbf{t}_\sigma := \left(\mathbf{s}_{V^{\otimes 2d},V^{*\otimes 2d}} \circ (\mathsf{p}_d \otimes \mathsf{p}_d^*)\right)(\widetilde{\mathbf{t}}_\sigma),$$

where $\mathbf{s}_{V^{\otimes 2d},V^{*\otimes 2d}}$ is the symmetry isomorphism, defined in (1). The action of the linear transformation $T^{\sigma_1}_{V,2d} \otimes T^{\sigma_2}_{V^*,2d}$ on $V^{\otimes 2d}\otimes V^{*\otimes 2d} \cong \mathrm{End}_{\mathbb{C}}(V^{\otimes 2d})$ is given by

$$T \mapsto T^{\sigma_1}_{V,2d} T T^{\pi\sigma_2^{-1}\pi^{-1}}_{V,2d} \text{ for every } T \in \mathrm{End}_{\mathbb{C}}(V^{\otimes 2d}),$$

where $\pi \in S_{2d}$ is defined by $\pi(i) = 2d+1-i$ for every $1 \le i \le 2d$. Thus from (20) it follows that

$$\begin{aligned}\mathbf{t}_\sigma &= \left(\frac{1}{d!2^d}\right)^2 \sum_{\sigma',\sigma''\in H_{2d}} \mathbf{t}_{\sigma'\sigma\sigma''} \qquad (21)\\ &= \frac{1}{2^d}\sum_{i_1,\ldots,i_{2d}\in\mathcal{I}_{m,n}} (-1)^{|i_1|+\cdots+|i_{2d}|+\epsilon\left(\sigma;\mathbf{e}_{i_1},\ldots,\mathbf{e}_{i_{2d}}\right)} y_{i_{2d-1},i_{2d}}\cdots y_{i_1,i_2}\otimes x_{i_{\sigma(1)},i_{\sigma(2)}}\cdots x_{i_{\sigma(2d-1)},i_{\sigma(2d)}}.\end{aligned}$$

Lemma 3.5. *Let $\sigma \in S_{2d}$, $\sigma' \in S_{2d'}$, and $\sigma'' \in S_{2d''}$ be such that $d' + d'' = d$ and*

$$\sigma(r) := \begin{cases}\sigma'(r) & \text{if } 1 \le r \le 2d',\\ \sigma''(r-2d') + 2d' & \text{if } 2d'+1 \le r \le 2d.\end{cases}$$

Then $\mathbf{t}_\sigma = \mathbf{t}_{\sigma'}\mathbf{t}_{\sigma''}$ as elements of the superalgebra $\mathscr{P}(W)\otimes\mathcal{S}(W)$.

Proof. Follows immediately from the explicit summation formula for $\mathbf{t}_\sigma$ given in (21). □

Proposition 3.6. *Let $\sigma \in S_{2d}$. Then the following statements hold.*

(i) $\mathbf{t}_\sigma = \mathbf{t}_{\sigma'\sigma\sigma''}$ *for every $\sigma', \sigma'' \in H_{2d}$.*

(ii) *There exist $\sigma', \sigma'' \in H_{2d}$ and integers $0 = d_0 < d_1 < \cdots < d_{r-1} < d_r = d$ such that*

$$\sigma'\sigma\sigma'' = \tau_{d_0,d_1}\tau_{d_1,d_2}\cdots\tau_{d_{r-1},d_r}, \tag{22}$$

where $\tau_{a,b}$ for $a < b$ denotes the cycle $(2a+2, 2a+4, \ldots, 2b-2, 2b)$.

Proof. (i) Follows from (21).
(ii) First we show that there exist integers $a_1, \ldots, a_d, b_1, \ldots, b_d$ and a permutation $\tau \in S_d$ such that

$$2s - 1 \leq a_s, b_s \leq 2s \text{ and } \sigma(a_s) = b_{\tau(s)} \text{ for } 1 \leq s \leq d.$$

To this end, we construct an undirected bipartite multigraph $\mathcal{G}$, with vertex set

$$\mathcal{V} := \{\mathbf{a}_1, \ldots, \mathbf{a}_d\} \cup \{\mathbf{b}_1, \ldots, \mathbf{b}_d\},$$

and $m_{s,s'} := \left|M_{s,s'}\right|$ edges between $\mathbf{a}_s$ and $\mathbf{b}_{s'}$, labelled by elements of the set $M_{s,s'}$, where

$$M_{s,s'} := \{\sigma(2s-1), \sigma(2s)\} \cap \{2s'-1, 2s'\} \text{ for all } 1 \leq s, s' \leq d.$$

Elements of $M_{s,s'}$ correspond to equalities of the form $\sigma(a_s) = b_{s'}$, where $2s - 1 \leq a_s \leq 2s$ and $2s' - 1 \leq b_{s'} \leq 2s'$. Every vertex in $\mathcal{G}$ has degree two, and therefore $\mathcal{G}$ is a union of disjoint cycles. It follows that $\mathcal{G}$ has a perfect matching, i.e., a set of d edges which do not have a vertex in common. Using this matching we define $\tau \in S_d$ so that the edges of the matching are $(\mathbf{a}_s, \mathbf{b}_{\tau(s)})$. The label on each edge $(\mathbf{a}_s, \mathbf{b}_{\tau(s)})$ of the matching is the corresponding $b_{\tau(s)}$, and $a_s := \sigma^{-1}\big(b_{\tau(s)}\big)$.

Now let $\sigma_1, \sigma_2 \in H_{2d}$ be defined by

$$\sigma_1(t) := \begin{cases} 2s-1 & \text{if } t = 2\tau(s) - 1 \text{ for } 1 \leq s \leq d, \\ 2s & \text{if } t = 2\tau(s) \text{ for } 1 \leq s \leq d. \end{cases} \quad \text{and} \quad \sigma_2 := \prod_{\substack{1 \leq s \leq d \\ a_s - b_{\tau(s)} \notin 2\mathbb{Z}}} (2s-1, 2s),$$

and set $\sigma_3 := \sigma_2 \sigma_1 \sigma$. It is easy to check that that $\sigma_3(a_s) = a_s$ for $1 \leq s \leq d$. Next set

$$\sigma_4 := \prod_{\substack{1 \leq s \leq d \\ a_s = 2s}} (2s-1, 2s) \text{ and } \sigma_5 := \sigma_4 \sigma_3 \sigma_4^{-1},$$

so that $\sigma_5(2s-1) = 2s-1$ for $1 \leq s \leq d$. Now let $\tau \in S_d$ be defined by $\tau(s) := \frac{1}{2}\sigma_5(2s)$ for $1 \leq s \leq d$. Choose $0 = d_0 < d_1 < \cdots < d_r = d$ and $\tau_\circ \in S_d$ such that

$$\tau_\circ \tau \tau_\circ^{-1} = \prod_{s=1}^{d} (d_{s-1} + 1, \ldots, d_s).$$

Finally, let $\sigma_\circ \in H_{2d}$ be defined by

$$\sigma_\circ(2s) := 2\tau_\circ(s) \text{ and } \sigma_\circ(2s-1) := 2\tau_\circ(s) - 1 \text{ for } 1 \leq s \leq d.$$

It is straightforward to check that $\sigma_\circ \sigma_5 \sigma_\circ^{-1} = \tau_{d_0,d_1} \cdots \tau_{d_{r-1},d_r}$, so that (22) holds for $\sigma' := \sigma_\circ \sigma_4 \sigma_2 \sigma_1$ and $\sigma'' := \sigma_4^{-1} \sigma_\circ^{-1}$. □

Lemma 3.7. *For every $d \geq 1$,*

$$\left(\mathscr{P}^d(W) \otimes \mathcal{S}^d(W)\right)^{\mathfrak{g}} = \mathrm{Span}_{\mathbb{C}} \left\{\mathbf{t}_\sigma \, : \, \sigma \in S_{2d}\right\}. \tag{23}$$

Furthermore, $(\mathscr{P}(W) \otimes \mathcal{S}(W))^{\mathfrak{g}} = \bigoplus_{d \geq 0} \left(\mathscr{P}^d(W) \otimes \mathcal{S}^d(W)\right)^{\mathfrak{g}}$.

Proof. For every $k,l \geq 0$, the action of $\mathfrak{g}$ on $\mathscr{P}(W)\otimes\mathcal{S}(W)$ leaves $\mathscr{P}^k(W)\otimes\mathcal{S}^l(W)$ invariant. It follows that $(\mathscr{P}(W)\otimes\mathcal{S}(W))^{\mathfrak{g}} = \bigoplus_{k,l\geq 0}\left(\mathscr{P}^k(W)\otimes\mathcal{S}^l(W)\right)^{\mathfrak{g}}$. By considering the action of the centre of $\mathfrak{g}$ we obtain that

$$\left(\mathscr{P}^k(W)\otimes\mathcal{S}^l(W)\right)^{\mathfrak{g}} = \{0\} \text{ unless } k=l. \tag{24}$$

Next we prove (23). Recall the definition of p_d and p_d^* from (20). It suffices to prove that the canonical projection

$$V^{*\otimes 2d}\otimes V^{\otimes 2d} \xrightarrow{\mathsf{p}_d^*\otimes\mathsf{p}_d} \mathcal{S}^d(W^*)\otimes\mathcal{S}^d(W) \tag{25}$$

is surjective on $\mathfrak{g}$-invariants. This follows from $\mathcal{S}^d(W^*)\otimes\mathcal{S}^d(W) = \left(V^{*\otimes 2d}\otimes V^{\otimes 2d}\right)^{H_{2d}\times H_{2d}}$ and the fact that $\mathsf{p}_d^*\otimes\mathsf{p}_d$ restricts to the identity map on $\mathcal{S}^d(W^*)\otimes\mathcal{S}^d(W)$. □

For every $i,j\in\mathcal{I}_{m,n}$, let $\varphi_{i,j}\in\mathscr{P}(W)\otimes\mathcal{S}(W)$ be defined by

$$\varphi_{i,j} := (-1)^{|i|\cdot|j|}\sum_{r\in\mathcal{I}_{m,n}}(-1)^{|r|}y_{r,j}\otimes x_{r,i}.$$

Note that

$$\mathsf{m}(\varphi_{i,j}) = -\check{\rho}(E_{i,j}). \tag{26}$$

Lemma 3.8. *Fix $d\geq 1$ and let $\sigma := (2,4,\dots,2d)\in S_{2d}$, that is, $\sigma(2r)=2r+2$ for $1\leq r\leq d-1$, $\sigma(2d)=2$, and $\sigma(2r-1)=2r-1$ for $1\leq r\leq d$. Then*

$$\mathbf{t}_\sigma = (-2)^d\,\widehat{\mathsf{s}}_d\big(\check{\rho}\big(\mathrm{str}\big(\mathbf{E}^d\big)\big)\big),$$

where $\mathbf{E}$ is the matrix defined in Lemma 3.1.

Proof. For $i_1,\dots,i_{2d}\in\mathcal{I}_{m,n}$, we set

$$\epsilon'(i_1,\dots,i_{2d}) := \sum_{s=1}^{2d}|i_s| + \sum_{s=3}^{2d}|i_2|\cdot|i_s| + \sum_{s=1}^{d-1}|i_{2s+1}|\cdot|i_{2s+2}|.$$

By a straightforward but tedious sign calculation, we obtain from (21) that

$$\begin{aligned}
\mathbf{t}_\sigma &= \frac{1}{2^d}\sum_{i_1,\dots,i_{2d}\in\mathcal{I}_{m,n}}(-1)^{\epsilon'(i_1,\dots,i_{2d})}y_{i_{2d-1},i_{2d}}\cdots y_{i_1,i_2}\otimes x_{i_1,i_{\sigma(2)}}\cdots x_{i_{2d-1},i_{\sigma(2d)}}\\
&= \frac{1}{2^d}\sum_{t_1,\dots,t_d\in\mathcal{I}_{m,n}}\left((-1)^{|t_1|+\sum_{s=1}^d|t_s|\cdot|t_{s+1}|}\prod_{s=1}^d\varphi_{t_{s+1},t_s}\right)
\end{aligned}$$

where $t_s := i_{2s}$ for $1\leq s\leq d$ and $t_{d+1} := i_2$. Using (11) and (26) we can write

$$\begin{aligned}
\widehat{\mathsf{s}}_d\big(\check{\rho}\big(\mathrm{str}\big(\mathbf{E}^d\big)\big)\big) &= \sum_{t_1,\dots,t_d\in\mathcal{I}_{m,n}}(-1)^{|t_1|+\sum_{s=1}^d|t_s|\cdot|t_{s+1}|}\,\widehat{\mathsf{s}}_d\left(\check{\rho}(E_{t_2,t_1})\cdots\check{\rho}(E_{t_1,t_d})\right)\\
&= \sum_{t_1,\dots,t_d\in\mathcal{I}_{m,n}}(-1)^{|t_1|+\sum_{s=1}^d|t_s|\cdot|t_{s+1}|}\,\widehat{\mathsf{s}}_1\left(\check{\rho}(E_{t_2,t_1})\right)\cdots\widehat{\mathsf{s}}_1\left(\check{\rho}(E_{t_1,t_d})\right)\\
&= (-2)^d\mathbf{t}_\sigma.
\end{aligned}$$

□

Let $V := \mathbb{C}^{m|n}$, $W := \mathcal{S}^2(V)$, $\mathfrak{g} := \mathfrak{gl}(V) = \mathfrak{gl}(m|n)$, and $\check{\rho}$ be as above. Our first main result is the following theorem.

Theorem 3.9. (Abstract Capelli Theorem for $W := \mathcal{S}^2(V)$.) *For every $d \geq 0$, we have*

$$\mathscr{P}\mathscr{D}^d(W)^{\mathfrak{g}} = \check{\rho}\left(\mathbf{Z}(\mathfrak{g}) \cap \mathbf{U}^d(\mathfrak{g})\right).$$

Proof. The inclusion $\supseteq$ is obvious by definition. The proof of the inclusion $\subseteq$ is by induction on d. The case $d = 0$ is obvious. Next assume that the statement holds for all $d' < d$. Fix $D \in \mathscr{P}\mathscr{D}^d(W)^{\mathfrak{g}}$ such that $\mathrm{ord}(D) = d$. It suffices to find $z_D \in \mathbf{Z}(\mathfrak{g}) \cap \mathbf{U}^d(\mathfrak{g})$ such that $\widehat{\mathsf{s}}_d\left(\check{\rho}(z_D) - D\right) = 0$.

Step 1. Choose $\varphi^D \in \mathscr{P}(W) \otimes \mathcal{S}(W)$ such that $\mathsf{m}(\varphi^D) = D$. We can write

$$\varphi^D = \varphi_0^D + \cdots + \varphi_d^D \text{ where } \varphi_r^D \in \mathscr{P}(W) \otimes \mathcal{S}^r(W) \text{ for } 0 \leq r \leq d,$$

and Lemma 3.7 implies that $\varphi_r^D \in (\mathscr{P}^r(W) \otimes \mathcal{S}^r(W))^{\mathfrak{g}}$ for $0 \leq r \leq d$. Furthermore,

$$\widehat{\mathsf{s}}_d(D) = \widehat{\mathsf{s}}_d\left(\mathsf{m}\left(\varphi^D\right)\right) = \widehat{\mathsf{s}}_d\left(\mathsf{m}\left(\varphi_d^D\right)\right) = \varphi_d^D.$$

By Lemma 3.7, φ_d^D is a linear combination of the tensors $\mathbf{t}_\sigma$ for $\sigma \in S_{2d}$. Thus to complete the proof, it suffices to find $z_\sigma \in \mathbf{Z}(\mathfrak{g}) \cap \mathbf{U}^d(\mathfrak{g})$ such that $\widehat{\mathsf{s}}_d\left(\check{\rho}(z_\sigma)\right) = \mathbf{t}_\sigma$.

Step 2. Fix $\sigma \in S_{2d}$ and let $0 = d_0 < d_1 \cdots < d_{r-1} < d_r = d$ be as in Proposition 3.6(ii). Set $\sigma_s := (2, \ldots, 2(d_{s+1} - d_s)) \in S_{2(d_{s+1}-d_s)}$ for $0 \leq s \leq r-1$. From Proposition 3.6(i) and Lemma 3.5 it follows that

$$\mathbf{t}_\sigma = \mathbf{t}_{\sigma_0} \cdots \mathbf{t}_{\sigma_{r-1}}.$$

Now set $z_s := (-\frac{1}{2})^{d_{s+1}-d_s} \operatorname{str}\left(\mathbf{E}^{d_{s+1}-d_s}\right)$ for $0 \leq s \leq r-1$, so that $z_s \in \mathbf{Z}(\mathfrak{g})$ by Lemma 3.1. By Lemma 3.8,

$$\widehat{\mathsf{s}}_{d_{s+1}-d_s}\left(\check{\rho}(z_s)\right) = \mathbf{t}_{\sigma_s} \text{ for } 0 \leq s \leq r-1. \tag{27}$$

From (27) it follows that

$$\begin{aligned}\widehat{\mathsf{s}}_d(\check{\rho}(z_0 \cdots z_{r-1})) &= \widehat{\mathsf{s}}_d(\check{\rho}(z_0) \cdots \check{\rho}(z_{r-1})) \\ &= \widehat{\mathsf{s}}_{d_1-d_0}(\check{\rho}(z_0)) \cdots \widehat{\mathsf{s}}_{d_r-d_{r-1}}(\check{\rho}(z_{r-1})) = \mathbf{t}_{\sigma_0} \cdots \mathbf{t}_{\sigma_{r-1}} = \mathbf{t}_\sigma,\end{aligned}$$

which, as mentioned in Step 1, completes the proof of the theorem. $\square$

4 The symmetric superpair $\left(\mathfrak{gl}(m|2n), \mathfrak{osp}(m|2n)\right)$

From now on, we assume that the odd part of V is even dimensional. In other words, we set $V := \mathbb{C}^{m|2n}$. We set $W := \mathcal{S}^2(V)$ and $\mathfrak{g} := \mathfrak{gl}(V) \cong \mathfrak{gl}(m|2n)$, as before. Let $\beta \in W^*$ be a nondegenerate, symmetric, even, bilinear form on V, so that for every homogeneous $v, v' \in V$ we have $\beta(v, v') = 0$ if $|v| \neq |v'|$, and $\beta(v', v) = (-1)^{|v| \cdot |v'|}\beta(v, v')$. Since $\beta \in W^*$ is even, we have

$$(\check{\rho}(x)\beta)(w) = -(-1)^{|x| \cdot |\beta|}\beta(\rho(x)w) = -\beta(\rho(x)w) \text{ for } x \in \mathfrak{g},\ w \in W.$$

Set $\mathfrak{k} := \mathfrak{osp}(V,\beta) := \{x \in \mathfrak{gl}(V) \,:\, \check{\rho}(x)\beta = 0\}$, so that

$$\mathfrak{k} = \left\{ x \in \mathfrak{g} \,:\, \beta(x\cdot v, v') + (-1)^{|x|\cdot|v|}\beta(v, x\cdot v') = 0 \text{ for } v, v' \in V \right\}.$$

In this section we prove that every irreducible $\mathfrak{g}$-submodule of $\mathscr{P}(W)$ contains a non-zero $\mathfrak{k}$-invariant vector. We remark that this statement does not follow from [1, Thm A].

Recall the decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ from Section 3. Let J_{2n} be the $2n \times 2n$ block-diagonal matrix defined by

$$J_{2n} := \mathrm{diag}(\underbrace{J, \dots, J}_{n \text{ times}}) \text{ where } J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let I_m denote the $m \times m$ identity matrix. Without loss of generality we can fix a homogeneous basis $\{\mathbf{e}_i \,:\, i \in \mathcal{I}_{m,2n}\}$ for $V \cong \mathbb{C}^{m|2n}$ such that

$$\left[\beta(\mathbf{e}_i, \mathbf{e}_j)\right]_{i,j\in\mathcal{I}_{m,2n}} = \begin{bmatrix} I_m & 0 \\ 0 & J_{2n} \end{bmatrix}. \tag{28}$$

Set $\mathfrak{a} := \left\{ h \in \mathfrak{h} \,:\, \varepsilon_{\overline{2k-1}}(h) = \varepsilon_{\overline{2k}}(h) \text{ for } 1 \le k \le n \right\}$ and

$$\mathfrak{t} := \left\{ h \in \mathfrak{h} \,:\, \varepsilon_k(h) = 0 \text{ for } 1 \le k \le m \text{ and } \varepsilon_{\overline{2l-1}}(h) = -\varepsilon_{\overline{2l}}(h) \text{ for } 1 \le l \le n \right\}.$$

Then $\mathfrak{t} = \mathfrak{k} \cap \mathfrak{h}$ and $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Set

$$\gamma_k := \varepsilon_k\big|_{\mathfrak{a}} \text{ for } 1 \le k \le m \text{ and } \gamma_{\overline{l}} := \varepsilon_{\overline{2l}}\big|_{\mathfrak{a}} \text{ for } 1 \le l \le n. \tag{29}$$

The restricted root system of $\mathfrak{g}$ corresponding to $\mathfrak{a}$ will be denoted by $\Sigma := \Sigma^+ \cup \Sigma^-$, where $\Sigma^+ := \Sigma^+_{\mathbf{0}} \cup \Sigma^+_{\mathbf{1}}$ is explicitly given by

$$\Sigma^+_{\mathbf{0}} := \left\{ \gamma_k - \gamma_l \right\}_{1 \le k < l \le m} \cup \left\{ \gamma_{\overline{k}} - \gamma_{\overline{l}} \right\}_{1 \le k < l \le n} \text{ and } \Sigma^+_{\mathbf{1}} := \left\{ \gamma_k - \gamma_{\overline{l}} \right\}_{1 \le k \le m,\, 1 \le l \le n}.$$

Furthermore, as usual $\Sigma^- = -\Sigma^+$.

For every $\gamma \in \Sigma$ we set $\mathfrak{g}_\gamma := \{x \in \mathfrak{g} \,:\, [h,x] = \gamma(h)x \text{ for every } h \in \mathfrak{a}\}$. Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{u}^+ \text{ where } \mathfrak{u}^\pm := \bigoplus_{\gamma\in\Sigma^\pm} \mathfrak{g}_\gamma \subseteq \mathfrak{n}^\pm. \tag{30}$$

Remark 4.1. Recall from Section 3 that using the basis $\{\mathbf{e}_i \,:\, i \in \mathcal{I}_{m,2n}\}$ of $V = \mathbb{C}^{m|2n}$, we can represent every element of $\mathfrak{g}$ as an $(m,2n)$-block matrix whose rows and columns are indexed by the elements of $\mathcal{I}_{m,2n}$. The following matrices form a spanning set of $\mathfrak{k}$.

(i) $E_{k,l} - E_{l,k}$ for $1 \le k \ne l \le m$,
(ii) $E_{\overline{2l-1},\overline{2k-1}} - E_{\overline{2k},\overline{2l}}$ for $1 \le k,l \le n$,
(iii) $E_{\overline{2l-1},\overline{2k}} + E_{\overline{2k-1},\overline{2l}}$ for $1 \le k,l \le n$,
(iv) $E_{\overline{2l},\overline{2k-1}} + E_{\overline{2k},\overline{2l-1}}$ for $1 \le k,l \le n$,
(v) $E_{k,\overline{2l-1}} + E_{\overline{2l},k}$ for $1 \le k \le m$ and $1 \le l \le n$,

(vi) $E_{k,\overline{2l}} - E_{\overline{2l-1},k}$ for $1 \leq k \leq m$ and $1 \leq l \leq n$.

Every irreducible finite dimensional representation of $\mathfrak{g}$ is a highest weight module. Unless stated otherwise, the highest weights that we consider will be with respect to the standard Borel subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$. In this case, the highest weight $\lambda \in \mathfrak{h}^*$ can be written as

$$\lambda = \sum_{k=1}^{m} \lambda_k \varepsilon_k + \sum_{l=1}^{2n} \lambda_{m+l} \varepsilon_{\overline{l}} \in \mathfrak{h}^*,$$

such that $\lambda_k - \lambda_{k+1} \in \mathbb{Z}_{\geq 0}$ for every $k \in \{1, \ldots, m-1\} \cup \{m+1, \ldots, m+2n-1\}$, where $\mathbb{Z}_{\geq 0} := \{x \in \mathbb{Z} : x \geq 0\}$.

Lemma 4.2. *Let U be an irreducible finite dimensional $\mathfrak{g}$-module. Then* $\dim U^{\mathfrak{k}} \leq 1$.

Proof. Let $\lambda \in \mathfrak{h}^*$ be the highest weight of U and set $U' := \bigoplus_{\mu \neq \lambda} U(\mu)$, where $U(\mu)$ is the μ-weight space of U. Clearly U' is a $\mathbb{Z}/2$-graded subspace of U of codimension one, which is invariant under the action of $\mathfrak{a} \oplus \mathfrak{u}^-$. Note also that $U^{\mathfrak{k}}$ is a $\mathbb{Z}/2$-graded subspace of U. Suppose that $\dim U^{\mathfrak{k}} \geq 2$, so that $U' \cap U^{\mathfrak{k}} \neq \{0\}$. Fix a non-zero homogeneous vector $u \in U' \cap U^{\mathfrak{k}}$. The PBW Theorem and the Iwasawa decomposition (30) imply that

$$U = \mathbf{U}(\mathfrak{g})u = \mathbf{U}(\mathfrak{a} \oplus \mathfrak{u}^-)\mathbf{U}(\mathfrak{k})u = \mathbf{U}(\mathfrak{a} \oplus \mathfrak{u}^-)u \subseteq U',$$

which is a contradiction. $\square$

A *partition* is a sequence $\flat := (\flat_1, \flat_2, \flat_3, \ldots)$ of integers satisfying $\flat_k \geq \flat_{k+1} \geq 0$ for every $k \geq 1$, and $\flat_k = 0$ for all but finitely many $k \geq 1$. The *size* of $\flat$ is defined to be $|\flat| := \sum_{k=1}^{\infty} \flat_k$. The *transpose* of $\flat$ is denoted by $\flat' := (\flat'_1, \flat'_2, \flat'_3 \ldots)$, where

$$\flat'_k := |\{l \geq 1 \ : \ \flat_l \geq k\}|.$$

Definition 4.3. A partition $(\flat_1, \flat_2, \flat_3, \ldots)$ that satisfies $\flat_{m+1} \leq n$ is called an $(m|n)$-*hook partition*. The set of $(m|n)$-hook partitions of size d will be denoted by $\mathrm{H}_{m,n,d}$, and we set $\mathrm{H}_{m,n} := \bigcup_{d=0}^{\infty} \mathrm{H}_{m,n,d}$.

We now define a map

$$\mathbf{\Gamma} : \mathrm{H}_{m,n} \to \mathfrak{a}^* \tag{31}$$

as follows. To every $\flat = (\flat_1, \flat_2, \flat_3, \ldots) \in \mathrm{H}_{m,n}$ we associate the element $\lambda = \mathbf{\Gamma}(\flat) \in \mathfrak{a}^*$ given by

$$\lambda := \sum_{k=1}^{m} 2\flat_k \gamma_k + \sum_{l=1}^{n} 2 \max\left\{\flat'_l - m, 0\right\} \gamma_{\overline{l}} \in \mathfrak{a}^*. \tag{32}$$

For integers $m, n, d \geq 0$, set

$$\mathrm{E}_{m,n,d} := \mathbf{\Gamma}(\mathrm{H}_{m,n,d}) := \left\{\lambda \in \mathfrak{a}^* \ : \ \lambda = \mathbf{\Gamma}(\flat) \text{ for some } \flat \in \mathrm{H}_{m,n,d}\right\}$$

and

$$\mathrm{E}_{m,n} := \bigcup_{d=0}^{\infty} \mathrm{E}_{m,n,d}. \tag{33}$$

Because of the decomposition $\mathfrak{h}=\mathfrak{a}\oplus\mathfrak{t}$, every $\lambda\in\mathfrak{a}^*$ can be extended trivially on $\mathfrak{t}$ to yield an element of $\mathfrak{h}^*$. It is straightforward to verify that for every $\lambda\in\mathrm{E}_{m,n}$, the above extension of λ to $\mathfrak{h}^*$ is the highest weight of an irreducible $\mathfrak{g}$-module. We will denote the latter highest weight module by V_λ. From [4, Thm 3.4] or [3] it follows that

$$\mathcal{S}^d(W)\cong\bigoplus_{\lambda\in\mathrm{E}_{m,n,d}}V_\lambda\quad\text{as }\mathfrak{g}\text{-modules.}\tag{34}$$

Remark 4.4. From (34) it follows that

$$\mathscr{P}^d(W)\cong\bigoplus_{\lambda\in\mathrm{E}_{m,n,d}}V_\lambda^*\cong\bigoplus_{\mu\in\mathrm{E}^*_{m,n,d}}V_\mu\tag{35}$$

where $\mathrm{E}^*_{m,n,d}$ is the set of all $\mu\in\mathfrak{a}^*$ of the form

$$\mu=-\sum_{i=1}^m2\max\{\flat_{m+1-i}-n,0\}\gamma_i-\sum_{j=1}^n2\flat'_{n+1-j}\gamma_{\bar{j}}\text{ for some }\flat\in\mathrm{H}_{m,n,d},\tag{36}$$

and V_μ denotes the $\mathfrak{g}$-module whose highest weight is the extension of μ trivially on $\mathfrak{t}$ to $\mathfrak{h}^*$. The decomposition (35) follows from the fact that for every $\lambda\in\mathrm{E}_{m,n,d}$, the contragredient module V_λ^* has highest weight equal to the extension to $\mathfrak{h}^*$ of $-\lambda^-$, where λ^- is the highest weight of V_λ with respect to the opposite Borel subalgebra $\mathfrak{b}^-:=\mathfrak{h}\oplus\mathfrak{n}^-$. From the calculation of λ^- given in [19, Thm 6.1] or [5, Sec. 2.4.1], it follows that the highest weight of V_λ^* is of the form (36).

Set

$$\mathrm{E}^*_{m,n}:=\bigcup_{d=0}^\infty\mathrm{E}^*_{m,n,d}.\tag{37}$$

By the $\mathfrak{g}$-equivariant isomorphism (9),

$$\mathscr{P}\mathscr{D}(W)^\mathfrak{g}\cong\big(\mathscr{P}(W)\otimes\mathcal{S}(W)\big)^\mathfrak{g}\cong\bigoplus_{\lambda,\mu\in\mathrm{E}^*_{m,n}}(V_\lambda\otimes V_\mu^*)^\mathfrak{g}\cong\bigoplus_{\lambda,\mu\in\mathrm{E}^*_{m,n}}\mathrm{Hom}_\mathfrak{g}(V_\mu,V_\lambda).\tag{38}$$

Every non-zero element of $\mathrm{Hom}_\mathfrak{g}(V_\mu,V_\lambda)$ should map $V_\mu^{\mathfrak{n}^+}$ to $V_\lambda^{\mathfrak{n}^+}$, and is uniquely determined by the image of a highest weight vector in $V_\mu^{\mathfrak{n}^+}$. But $\dim(V_\lambda^{\mathfrak{n}^+})=\dim(V_\mu^{\mathfrak{n}^+})=1$, so that $\dim\left(\mathrm{Hom}_\mathfrak{g}(V_\lambda,V_\mu)\right)\leq1$ with equality if and only if $\lambda=\mu$.

Definition 4.5. For $\lambda\in\mathrm{E}^*_{m,n}$, let $D_\lambda\in\mathscr{P}\mathscr{D}(W)^\mathfrak{g}$ be the $\mathfrak{g}$-invariant differential operator that corresponds via the sequence of isomorphisms (38) to $1_{V_\lambda}\in\mathrm{End}_\mathbb{C}(V_\lambda)$.

From (38) it follows that $\{D_\lambda:\lambda\in\mathrm{E}^*_{m,n}\}$ is a basis for $\mathscr{P}\mathscr{D}(W)^\mathfrak{g}$. In the purely even case (that is, when $n=0$), the latter basis is sometimes called the *Capelli basis* of $\mathscr{P}\mathscr{D}(W)^\mathfrak{g}$.

Let $\beta\in W^*$ be the symmetric bilinear form defining $\mathfrak{k}$, as at the beginning of Section 4. Let $\mathsf{h}_\beta:\mathcal{S}(W)\to\mathbb{C}$ be the extension of β, defined as in (6). Set

$$\widetilde{\mathsf{h}}_\beta:\mathscr{P}(W)\otimes\mathcal{S}(W)\to\mathscr{P}(W)\ ,\ a\otimes b\mapsto\mathsf{h}_\beta(b)a.\tag{39}$$

Recall the map $\mathsf{m}:\mathscr{P}(W)\otimes\mathcal{S}(W)\to\mathscr{P}\mathscr{D}(W)$ defined in (9).

Proposition 4.6. *Let $\lambda \in \mathrm{E}^*_{m,n}$. Then $\dim V_\lambda^{\mathfrak{k}} = 1$, and the vector*

$$\mathbf{d}_\lambda := (\widetilde{\mathsf{h}}_\beta \circ \mathsf{m}^{-1})(D_\lambda).$$

is a non-zero $\mathfrak{k}$-invariant vector of V_λ.

Proof. Since $\beta \in (W^*)^{\mathfrak{k}}$, the map h_β is $\mathfrak{k}$-equivariant. It follows that $\widetilde{\mathsf{h}}_\beta = 1_{\mathscr{P}(W)} \otimes \mathsf{h}_\beta$ is $\mathfrak{k}$-equivariant, and therefore $\mathbf{d}_\lambda \in V_\lambda^{\mathfrak{k}}$. By Lemma 4.2, it suffices to prove that $\mathbf{d}_\lambda \neq 0$. Fix $d \geq 0$ such that $\lambda \in \mathrm{E}^*_{m,n,d}$, so that $V_\lambda \in \mathscr{P}^d(W)$ and $V_\lambda^* \subseteq \mathcal{S}^d(W)$.

Step 1. The linear map $\mathsf{h}_\beta\big|_{V_\lambda^*} : V_\lambda^* \to \mathbb{C}$ can be represented by evaluation at a vector $v_\circ \in V_\lambda$, that is, $\mathsf{h}_\beta(v^*) := \langle v^*, v_\circ\rangle$ for every $v^* \in V_\lambda^*$.

Let $\iota_\lambda : V_\lambda \otimes V_\lambda^* \to \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_\lambda)$ be the isomorphism (2). For every $v \otimes v^* \in V_\lambda \otimes V_\lambda^*$,

$$\widetilde{\mathsf{h}}_\beta \circ \iota_\lambda^{-1}(T_{v\otimes v^*}) = \widetilde{\mathsf{h}}_\beta(v \otimes v^*) = \mathsf{h}_\beta(v^*)v = \langle v^*, v_\circ\rangle v = T_{v\otimes v^*}(v_\circ).$$

It follows that $\widetilde{\mathsf{h}}_\beta \circ \iota_\lambda^{-1}(T) = Tv_\circ$ for every $T \in \mathrm{End}_{\mathbb{C}}(V_\lambda)$. In particular,

$$\mathbf{d}_\lambda = \widetilde{\mathsf{h}}_\beta \circ \iota_\lambda^{-1}(1_\lambda) = v_\circ,$$

so that to complete the proof we need to show that $v_\circ \neq 0$, or equivalently, that $\mathsf{h}_\beta\big|_{V_\lambda^*} \neq 0$.

Step 2. Let $\{x_{i,j} \;:\; i,j \in \mathcal{I}_{m,2n}\}$ be the generating set of $\mathcal{S}(W)$ that is defined in (15). We define the superderivations $\mathsf{D}_{i,j}$, for $i,j \in \mathcal{I}_{m,2n}$, as in (17). Assume that $\mathsf{h}_\beta\big|_{V_\lambda^*} = 0$. We will reach a contradiction in Step 3. Fix $a \in V_\lambda^* \subseteq \mathcal{S}^d(W)$. We claim that

$$\mathsf{h}_\beta\left(\mathsf{D}_{j_1,i_1} \cdots \mathsf{D}_{j_k,i_k}(a)\right) = 0 \text{ for every } k \geq 0,\ i_1, j_1, \ldots, i_k, j_k \in \mathcal{I}_{m,2n}, \tag{40}$$

where $\mathsf{D}_{i,j}$ is defined as in (17). The proof of (40) is by induction on k. For $k = 0$, it follows from the assumption that $\mathsf{h}_\beta\big|_{V_\lambda^*} = 0$. Next assume $k = 1$. Set

$$i_1' := \begin{cases} i_1 & \text{if } |i_1| = \overline{\mathbf{0}}, \\ \overline{2k} & \text{if } i_1 = \overline{2k-1} \text{ where } 1 \leq k \leq n, \\ \overline{2k-1} & \text{if } i_1 = \overline{2k} \text{ where } 1 \leq k \leq n, \end{cases} \tag{41}$$

Then $\rho\big(E_{i_1',j_1}\big)a \in V_\lambda$, so that $\mathsf{h}_\beta\big(\rho\big(E_{i_1',j_1}\big)a\big) = 0$. But from (16) it follows that

$$\begin{aligned} \mathsf{h}_\beta\big(\rho\big(E_{i_1',j_1}\big)a\big) &= \sum_{r \in \mathcal{I}_{m,2n}} \mathsf{h}_\beta\big(x_{i_1',r}\big)\mathsf{h}_\beta\big(\mathsf{D}_{j_1,r}(a)\big) \\ &= \sum_{r \in \mathcal{I}_{m,2n}} \beta(\mathbf{e}_{i_1'}, \mathbf{e}_r)\mathsf{h}_\beta\big(\mathsf{D}_{j_1,r}(a)\big) = \pm\mathsf{h}_\beta\big(\mathsf{D}_{j_1,i_1}(a)\big), \end{aligned}$$

so that $\mathsf{h}_\beta(\mathsf{D}_{j_1,i_1}(a)) = 0$.

Finally, assume that $k > 1$. We define $i'_1, \dots, i'_k \in \mathcal{I}_{m,2n}$ from $i_1, \dots, i_k$ according to (41). Then $\rho\big(E_{i'_1,j_1}\big)\cdots\rho\big(E_{i'_k,j_k}\big)a \in V_\lambda$, so that $\mathsf{h}_\beta\big(\rho\big(E_{i'_1,j_1}\big)\cdots\rho\big(E_{i'_k,j_k}\big)a\big) = 0$. However, we can write

$$\rho\big(E_{i'_1,j_1}\big)\cdots\rho\big(E_{i'_k,j_k}\big)a = \sum_{r_1,\dots,r_k\in\mathcal{I}_{m,2n}} x_{i'_1,r_1}\cdots x_{i'_k,r_k}\mathsf{D}_{j_1,r_1}\cdots\mathsf{D}_{j_k,r_k}(a) + R_a, \tag{42}$$

where R_a is a sum of elements of $\mathcal{S}(W)$ of the form $b\mathsf{D}_{s_1,t_1}\cdots\mathsf{D}_{s_\ell,t_\ell}(a)$, with $b \in \mathcal{S}(W)$ and $\ell < k$. Since h_β is an algebra homomorphism, the induction hypothesis implies that $\mathsf{h}_\beta(R_a) = 0$. Therefore (42) implies that

$$\begin{aligned}\mathsf{h}_\beta\big(\rho\big(E_{i'_1,j_1}\big)\cdots\rho\big(E_{i'_k,j_k}\big)a\big) &= \sum_{r_1,\dots,r_k\in\mathcal{I}_{m,2n}} \mathsf{h}_\beta\big(x_{i'_1,r_1}\big)\cdots\mathsf{h}_\beta\big(x_{i'_k,r_k}\big)\mathsf{h}_\beta\big(\mathsf{D}_{j_1,r_1}\cdots\mathsf{D}_{j_k,r_k}(a)\big)\\ &= \pm\mathsf{h}_\beta\big(\mathsf{D}_{j_1,i_1}\cdots\mathsf{D}_{j_k,i_k}(a)\big),\end{aligned}$$

so that $\mathsf{h}_\beta\big(\mathsf{D}_{j_1,i_1}\cdots\mathsf{D}_{j_k,i_k}(a)\big) = 0$.

Step 3. Set $\mathcal{J} := \{(k,\bar{l}) : 1 \le k \le m \text{ and } 1 \le l \le 2n\} \subset \mathcal{I}_{m,2n}\times\mathcal{I}_{m,2n}$. Every $a \in \mathcal{S}(W)$ can be written as

$$a = \sum_{S\subseteq\mathcal{J}} a_S x_S$$

where $a_S \in \mathcal{S}(W_{\overline{\mathbf{0}}})$ and $x_S := \prod_{(i,j)\in S} x_{i,j}$ for each $S \subseteq \mathcal{J}$. Now fix $S \subseteq \mathcal{J}$ and set $\mathsf{D}_S := \prod_{(i,j)\in S}\mathsf{D}_{i,j}$, so that $\mathsf{D}_S(a) = \pm a_S$. From (40) it follows that

$$\mathsf{h}_\beta\big(\mathsf{D}_{j_1,i_1}\cdots\mathsf{D}_{j_k,i_k})(a_S)\big) = 0 \text{ for every } k \ge 0,\ i_1,\dots,i_k,j_1,\dots,j_k \in \{1,\dots,m\}. \tag{43}$$

But $\mathcal{S}(W_{\overline{\mathbf{0}}}) \cong \mathscr{P}(W^*_{\overline{\mathbf{0}}})$, and therefore (43) means that the multi-variable polynomial $a_S \in \mathscr{P}(W^*_{\overline{\mathbf{0}}})$ and all of its partial derivatives vanish at a fixed point $\beta_{\overline{\mathbf{0}}} := \beta\big|_{W_{\overline{\mathbf{0}}}} \in W^*_{\overline{\mathbf{0}}}$. Consequently, $a_S = 0$. As $S \subseteq \mathcal{J}$ is arbitrary, it follows that $a = 0$. $\square$

Remark 4.7. A slight modification of the proof of Proposition 4.6 shows that for every $\lambda \in \mathrm{E}_{m,n}$ we have $\dim V_\lambda^{\mathfrak{k}} = 1$ as well. To this end, instead of $\beta \in W^*$ we use the $\mathfrak{k}$-invariant vector

$$\beta^* := -\frac{1}{4}(x_{1,1} + \cdots + x_{m,m}) + \frac{1}{2}(x_{\overline{1},\overline{2}} + \cdots + x_{\overline{2n-1},\overline{2n}}) \in W.$$

5 The eigenvalue and spherical polynomials c_λ and d_λ

As in the previous section, we set $V := \mathbb{C}^{m|2n}$, so that $\mathfrak{g} := \mathfrak{gl}(V) = \mathfrak{gl}(m|2n)$. Recall the decompostion of $\mathscr{P}(W)$ into irreducible $\mathfrak{g}$-modules given in (35) and (37), that is,

$$\mathscr{P}(W) = \bigoplus_{\lambda\in\mathrm{E}^*_{m,n}} V_\lambda.$$

For $\lambda \in \mathrm{E}^*_{m,n}$, let $D_\lambda \in \mathscr{PD}(W)^{\mathfrak{g}}$ be the $\mathfrak{g}$-invariant differential operator as in Definition 4.5. Since the decomposition (35) is multiplicity-free, for every $\mu \in \mathrm{E}^*_{m,n}$ the map

$$D_\lambda : V_\mu \to V_\mu$$

is multiplication by a scalar $c_\lambda(\mu) \in \mathbb{C}$.

Fix $d \geq 0$ such that $\lambda \in \mathrm{E}^*_{m,n,d}$, so that $V_\lambda \subseteq \mathscr{P}^d(W)$. By Theorem 3.9 we can choose $z_\lambda \in \mathbf{Z}(\mathfrak{g}) \cap \mathbf{U}^d(\mathfrak{g})$ such that

$$\check{\rho}(z_\lambda) = D_\lambda. \tag{44}$$

Next choose $z_{\lambda,\mathfrak{k}} \in \mathfrak{k}\mathbf{U}(\mathfrak{g}) \cap \mathbf{U}^d(\mathfrak{g})$, $z_{\lambda,\mathfrak{u}^+} \in \mathbf{U}(\mathfrak{g})\mathfrak{u}^+ \cap \mathbf{U}^d(\mathfrak{g})$, and $z_{\lambda,\mathfrak{a}} \in \mathbf{U}(\mathfrak{a}) \cap \mathbf{U}^d(\mathfrak{g})$ such that

$$z_\lambda = z_{\lambda,\mathfrak{k}} + z_{\lambda,\mathfrak{u}^+} + z_{\lambda,\mathfrak{a}}, \tag{45}$$

according to the decomposition $\mathbf{U}(\mathfrak{g}) = (\mathfrak{k}\mathbf{U}(\mathfrak{g}) + \mathbf{U}(\mathfrak{g})\mathfrak{u}^+) \oplus \mathbf{U}(\mathfrak{a})$. For $\mu \in \mathfrak{a}^*$ let

$$\mathsf{h}_\mu : \mathbf{U}(\mathfrak{a}) \cong \mathcal{S}(\mathfrak{a}) \to \mathbb{C}$$

be the extension of μ defined as in (6).

Lemma 5.1. *$c_\lambda(\mu) = \mathsf{h}_\mu(z_{\lambda,\mathfrak{a}})$ for every $\lambda, \mu \in \mathrm{E}^*_{m,n}$. In particular, $c_\lambda \in \mathscr{P}(\mathfrak{a}^*)$.*

Proof. Let v_μ denote a highest weight vector of $V_\mu \subseteq \mathscr{P}(W)$. By Remark 4.7, the contragredient module $V_\mu^* \subseteq \mathcal{S}(W)$ contains a non-zero $\mathfrak{k}$-invariant vector $v_\circ^*$. From the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{u}^+$ and the PBW Theorem it follows that $V_\mu = \mathbf{U}(\mathfrak{k})v_\mu$. This means that if $\langle v_\circ^*, v_\mu\rangle = 0$, then $\langle v_\circ^*, V_\mu\rangle = 0$, which is a contradiction. Thus $\langle v_\circ^*, v_\mu\rangle \neq 0$, and

$$\begin{aligned} c_\lambda(\mu)\langle v_\circ^*, v_\mu\rangle &= \langle v_\circ^*, D_\lambda v_\mu\rangle = \langle v_\circ^*, \check{\rho}(z_\lambda)v_\mu\rangle \\ &= \langle v_\circ^*, (\check{\rho}(z_{\lambda,\mathfrak{k}}) + \check{\rho}(z_{\lambda,\mathfrak{u}^+}) + \check{\rho}(z_{\lambda,\mathfrak{a}}))v_\mu\rangle = \langle v_\circ^*, \check{\rho}(z_{\lambda,\mathfrak{a}})v_\mu\rangle = \mathsf{h}_\mu(z_{\lambda,\mathfrak{a}})\langle v_\circ^*, v_\mu\rangle. \end{aligned}$$

It follows that $c_\lambda(\mu) = \mathsf{h}_\mu(z_{\lambda,\mathfrak{a}})$. ☐

Definition 5.2. For every $\lambda \in \mathrm{E}^*_{m,n}$, the polynomial $c_\lambda \in \mathscr{P}(\mathfrak{a}^*)$ is called the *eigenvalue polynomial* associated to λ.

Remark 5.3. Note that by Theorem 3.9, if $\lambda \in \mathrm{E}^*_{m,n,d}$ then $\deg(c_\lambda) \leq d$.

Lemma 5.4. *Let $\lambda \in \mathrm{E}^*_{m,n,d}$. Then $c_\lambda(\lambda) = d!$ and $c_\lambda(\mu) = 0$ for all other μ in $\bigcup_{d'=0}^d \mathrm{E}^*_{m,n,d'}$.*

Proof. Note that $D_\lambda \in \mathsf{m}(V_\lambda \otimes V_\lambda^*) \subseteq \mathsf{m}\left(\mathscr{P}^d(W) \otimes \mathcal{S}^d(W)\right)$ where m is the map defined in (9). Thus if $d' < d$ then $D_\lambda \mathscr{P}^{d'}(W) = \{0\}$. Next assume $d' = d$. Then the map

$$\mathscr{P}\mathscr{D}(W) \otimes \mathscr{P}(W) \to \mathscr{P}(W)\ ,\ D \otimes p \mapsto Dp \tag{46}$$

is $\mathfrak{g}$-equivariant, and since $D_\lambda \in \mathsf{m}(V_\lambda \otimes V_\lambda^*)$, the map (46) restricts to a $\mathfrak{g}$-equivariant map $V_\mu \to V_\lambda$ given by $p \mapsto D_\lambda p$. By Schur's Lemma, when $\lambda \neq \mu$, the latter map should be zero.

Finally, to prove that $\mathbf{c}_\lambda(\lambda) = d!$, we consider the bilinear form

$$\boldsymbol{\beta} : \mathscr{P}^d(W) \times \mathcal{S}^d(W) \to \mathbb{C}\ ,\ \boldsymbol{\beta}(a, b) := \partial_b a. \tag{47}$$

By a straightforward calculation one can verify that $\boldsymbol{\beta}(a, b) = d!\langle a, b\rangle$, where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between $\mathscr{P}^d(W) \cong \mathcal{S}^d(W)^*$ and $\mathcal{S}^d(W)$. Next we choose a basis $v_1, \ldots, v_t$ for V_λ. Let $v_1^*, \ldots, v_t^*$ be the corresponding dual basis of V_λ^*. From (47) it follows that

$$D_\lambda v_k = \mathsf{m}(\sum_{l=1}^t v_l \otimes v_l^*)v_k = \sum_{i=1}^t v_l \partial_{v_l^*}(v_k) = d!v_k$$

for every $1 \leq k \leq t$. ☐

Let $\beta^* \in W$ be the $\mathfrak{k}$-invariant vector that is given in Remark 4.7, and set

$$\iota_{\mathfrak{a}} : \mathfrak{a} \to W \ , \ \iota_{\mathfrak{a}}(h) := \rho(h)\beta^*.$$

By duality, the map $\iota_{\mathfrak{a}}$ results in a homomorphism of superalgebras

$$\iota_{\mathfrak{a}}^* : \mathscr{P}(W) \cong \mathcal{S}(W^*) \to \mathcal{S}(\mathfrak{a}^*) \cong \mathscr{P}(\mathfrak{a}), \tag{48}$$

which is defined uniquely by the relation $\langle \iota_{\mathfrak{a}}^*(w^*), h\rangle = \langle w^*, \iota_{\mathfrak{a}}(h)\rangle$ for $w^* \in W^*$, $h \in \mathfrak{a}$.

Let κ be the supertrace form on $\mathfrak{g}$ defined as in (13). The restriction of κ to $\mathfrak{a}$ is a non-degenerate symmetric bilinear form and yields a canonical isomorphism $\mathsf{j} : \mathfrak{a}^* \to \mathfrak{a}$ which is defined as follows. For every $\xi \in \mathfrak{a}^*$,

$$\mathsf{j}(\xi) := x_\xi \text{ if and only if } \kappa(\,\cdot\,, x_\xi) = \xi.$$

Let $\mathsf{j}^* : \mathscr{P}(\mathfrak{a}) \to \mathscr{P}(\mathfrak{a}^*)$ be defined by $\mathsf{j}^*(p) := p \circ \mathsf{j}$ for every $p \in \mathscr{P}(\mathfrak{a})$. For $\lambda \in \mathrm{E}^*_{m,n}$, let $\mathbf{d}_\lambda \in \mathscr{P}(W)^{\mathfrak{k}}$ be the $\mathfrak{k}$-invariant vector in V_λ as in Proposition 4.6.

Definition 5.5. For $\lambda \in \mathrm{E}^*_{m,n,d}$, we define the *spherical polynomial* d_λ to be

$$d_\lambda := \mathsf{j}^* \circ \iota_{\mathfrak{a}}^*(\mathbf{d}_\lambda) \in \mathscr{P}^d(\mathfrak{a}^*).$$

Proposition 5.6. *The restriction of $\iota_{\mathfrak{a}}^*$ to $\mathscr{P}(W)^{\mathfrak{k}}$ is an injection. In particular, $d_\lambda \neq 0$ for every λ.*

Proof. The proof of Proposition 5.6 will be given in Section A. ☐

Recall the definitions of $\widetilde{\mathsf{h}}_\beta$ and $z_{\lambda,\mathfrak{a}}$ from (39) and (45).

Lemma 5.7. *Let $\lambda \in \mathrm{E}^*_{m,n,d}$, and let $z_{\lambda,\mathfrak{a}}$ be defined as in* (45)*. Then*

$$d_\lambda = \big(\mathsf{j}^* \circ \iota_{\mathfrak{a}}^* \circ \widetilde{\mathsf{h}}_\beta \circ \widehat{\mathsf{s}}_d\big)(\check{\rho}(z_{\lambda,\mathfrak{a}})).$$

Proof. Let D_λ be the $\mathfrak{g}$-invariant differential operator as in Definition 4.5. Since we have $D_\lambda \in \mathsf{m}(\mathscr{P}^d(W) \otimes \mathcal{S}^d(W))$, from (10) we obtain

$$\mathbf{d}_\lambda = \widetilde{\mathsf{h}}_\beta(\mathsf{m}^{-1}(D_\lambda)) = \widetilde{\mathsf{h}}_\beta(\widehat{\mathsf{s}}_d(D_\lambda)). \tag{49}$$

From (49) it follows that

$$d_\lambda = \big(\mathsf{j}^* \circ \iota_{\mathfrak{a}}^* \circ \widetilde{\mathsf{h}}_\beta \circ \widehat{\mathsf{s}}_d\big)(\check{\rho}(z_\lambda)).$$

By the decomposition (45) and the fact that $z_{\lambda,\mathfrak{k}}, z_{\lambda,\mathfrak{u}^+}, z_{\lambda,\mathfrak{a}} \in \mathbf{U}^d(\mathfrak{g})$, it is enough to prove that if $x \in \mathfrak{k}\mathbf{U}^{d-1}(\mathfrak{g})$ or $x \in \mathbf{U}^{d-1}(\mathfrak{g})\mathfrak{u}^+$, then

$$\big(\iota_{\mathfrak{a}}^* \circ \widetilde{\mathsf{h}}_\beta \circ \widehat{\mathsf{s}}_d\big)(\check{\rho}(x)) = 0. \tag{50}$$

First we prove (50) for $x \in \mathfrak{k}\mathbf{U}^{d-1}(\mathfrak{g})$. By the PBW Theorem there exist $x^\circ, x^- \in \mathbf{U}(\mathfrak{g})$ such that such that

(i) $x = x^\circ + x^-$,
(ii) x° is a sum of monomials of the form $x_1 \cdots x_d$ where $x_1 \in \mathfrak{k}$ and $x_2, \ldots, x_d \in \mathfrak{g}$,
(iii) $x^- \in \mathbf{U}^{d-1}(\mathfrak{g})$.

Recall that $\mathrm{ord}(D)$ denotes the order of a differential operator $D \in \mathscr{PD}(W)$. Since we have $\mathrm{ord}(\check{\rho}(x)) \leq 1$ for every $x \in \mathfrak{g}$, from (11) it follows that $\widehat{\mathsf{s}}_d(\check{\rho}(x^-)) = 0$ and

$$\widehat{\mathsf{s}}_d(\check{\rho}(x_1 \cdots x_d)) = \widehat{\mathsf{s}}_1(\check{\rho}(x_1)) \cdots \widehat{\mathsf{s}}_1(\check{\rho}(x_d)). \tag{51}$$

Since $\iota_{\mathfrak{a}}^*$ and $\widetilde{\mathsf{h}}_\beta$ are homomorphisms of superalgebras, from (51) it follows that in order to prove (50), it is enough to verify that

$$\left(\iota_{\mathfrak{a}}^* \circ \widetilde{\mathsf{h}}_\beta \circ \widehat{\mathsf{s}}_1\right)(\check{\rho}(x_1)) = 0 \text{ for every } x_1 \in \mathfrak{k}. \tag{52}$$

To verify (52), we use the generators $\{x_{i,j} \,:\, i,j \in \mathcal{I}_{m,2n}\}$ and $\{y_{i,j} \,:\, i,j \in \mathcal{I}_{m,2n}\}$ defined as in (15). From (18) it follows that

$$\widehat{\mathsf{s}}_1(\check{\rho}(E_{i,j})) = -(-1)^{|i|\cdot|j|} \sum_{r \in \mathcal{I}_{m,2n}} (-1)^{|r|} y_{r,j} x_{r,i} \text{ for } i,j \in \mathcal{I}_{m,2n}, \tag{53}$$

and consequently,

$$\left(\iota_{\mathfrak{a}}^* \circ \widetilde{\mathsf{h}}_\beta \circ \widehat{\mathsf{s}}_1\right)(\check{\rho}(E_{i,j})) = \begin{cases} -\iota_{\mathfrak{a}}^*(y_{i,j}) & \text{if } |i| = |j| = \overline{\mathbf{0}}, \\ \iota_{\mathfrak{a}}^*(y_{\overline{2k},j}) & \text{if } i = \overline{2k-1} \text{ for } 1 \leq k \leq n, \text{ and } |j| = \overline{\mathbf{1}}, \\ -\iota_{\mathfrak{a}}^*(y_{\overline{2k-1},j}) & \text{if } i = \overline{2k} \text{ for } 1 \leq k \leq n, \text{ and } |j| = \overline{\mathbf{1}}, \\ 0 & \text{if } |i| \neq |j|. \end{cases}$$

If $x_1 \in \mathfrak{k}_{\overline{\mathbf{1}}}$, then x_1 is a linear combination of the $E_{i,j}$ such that $|i| \neq |j|$, and therefore $\left(\iota_{\mathfrak{a}}^* \circ \widetilde{\mathsf{h}}_\beta \circ \widehat{\mathsf{s}}_1\right)(\check{\rho}(x_1)) = 0$. If $x_1 \in \mathfrak{k}_{\overline{\mathbf{0}}}$, then x_1 is a linear combination of elements of the cases (i)–(iv) in Remark 4.1, and in each case, we can verify that $\left(\iota_{\mathfrak{a}}^* \circ \widetilde{\mathsf{h}}_\beta \circ \widehat{\mathsf{s}}_1\right)(\check{\rho}(x_1)) = 0$.

The proof of (50) for $x \in \mathbf{U}(\mathfrak{g})^{d-1}\mathfrak{u}^+$ is similar. As in the case $x \in \mathfrak{k}\mathbf{U}(\mathfrak{g})$, the proof can be reduced to showing that

$$\left(\iota_{\mathfrak{a}}^* \circ \widetilde{\mathsf{h}}_\beta \circ \widehat{\mathsf{s}}_1\right)(\check{\rho}(x)) = 0 \text{ for every } x \in \mathfrak{u}^+. \tag{54}$$

It is easy to verify directly that $\mathfrak{u}^+$ is spanned by the $E_{i,j}$ for i,j satisfying at least one of the following conditions.

(i) $1 \leq i < j \leq m$,

(ii) $i = \overline{k}$ and $j = \overline{l}$ where $1 \leq k < l \leq 2n$ and $(k,l) \notin \{(2t-1, 2t) \,:\, 1 \leq t \leq n\}$.

Therefore (54) follows from the above calculation of $\left(\iota_{\mathfrak{a}}^* \circ \widetilde{\mathsf{h}}_\beta \circ \widehat{\mathsf{s}}_1\right)(\check{\rho}(E_{i,j}))$, together with the fact that $\iota_{\mathfrak{a}}^*(y_{i,j}) = 0$ unless $i = j \in \{1, \ldots, m\}$, or $i = \overline{2k-1}$ and $j = \overline{2k}$ where $1 \leq k \leq n$, or $i = \overline{2k}$ and $j = \overline{2k-1}$ where $1 \leq k \leq n$. $\square$

For any polynomial $p \in \mathscr{P}(\mathfrak{a}^*)$, we write $\overline{p}$ for the homogeneous part of highest degree of p. Recall that if $\lambda \in \mathrm{E}_{m,n}^*$, then c_λ denotes the eigenvalue polynomial, as in Definition 5.2, and d_λ denotes the spherical polynomial, as in Definition 5.5. We have the following theorem.

Theorem 5.8. *For every $\lambda \in \mathrm{E}_{m,n}^*$, we have $\overline{c}_\lambda = d_\lambda$.*

Proof. Let γ_i be the basis of $\mathfrak{a}^*$ defined in (29), and let $\{h_i : i \in \mathcal{I}_{m,n}\}$ be the dual basis of $\mathfrak{a}$. Assume that $\lambda \in \mathrm{E}^*_{m,n,d}$, so that $V_\lambda \subset \mathscr{P}^d(W)$. Recall the definition of $z_{\lambda,\mathfrak{a}}$ from (45). By the PBW Theorem, we can write

$$z_{\lambda,\mathfrak{a}} = \sum_{k_1,\ldots,k_m,l_{\overline{1}},\ldots,l_{\overline{n}}\geq 0} u_{k_1,\ldots,k_m,l_{\overline{1}},\ldots,l_{\overline{n}}} h_1^{k_1}\cdots h_m^{k_m} h_{\overline{1}}^{l_{\overline{1}}}\cdots h_{\overline{n}}^{l_{\overline{n}}},$$

where only finitely many of the scalar coefficients $u_{k_1,\ldots,k_m,l_{\overline{1}},\ldots,l_{\overline{n}}}$ are nonzero. Now consider the polynomial $p_\lambda = p_\lambda(t_1,\ldots,t_{m+n})$ in $m+n$ variables $t_1,\ldots,t_{m+n}$, defined by

$$p_\lambda(t_1,\ldots,t_{m+n}) := \sum_{k_1,\ldots,k_m,l_{\overline{1}},\ldots,l_{\overline{n}}\geq 0} u_{k_1,\ldots,k_m,l_{\overline{1}},\ldots,l_{\overline{n}}} t_1^{k_1}\cdots t_m^{k_m} t_{m+1}^{l_{\overline{1}}}\cdots t_{m+n}^{l_{\overline{n}}},$$

so that $z_{\lambda,\mathfrak{a}} = p_\lambda\left(h_1,\ldots,h_m,h_{\overline{1}},\ldots,h_{\overline{n}}\right)$. We first note that $\deg(p_\lambda) = d$. Indeed since $z_{\lambda,\mathfrak{a}} \in \mathbf{U}^d(\mathfrak{g})$, it follows that $\deg(p_\lambda) \leq d$. If $\deg(p_\lambda) < d$, then $\operatorname{ord}(\check{\rho}(z_{\lambda,\mathfrak{a}})) < d$, and Lemma 5.7 implies that $d_\lambda = 0$, which contradicts Proposition 5.6.

Fix $\xi := \sum_{i\in\mathcal{I}_{m,n}} a_i\gamma_i \in \mathfrak{a}^*$. Lemma 5.1 implies that

$$c_\lambda(\xi) = p_\lambda(\xi(h_1),\ldots,\xi(h_m),\xi(h_{\overline{1}}),\ldots,\xi(h_{\overline{n}})) = p_\lambda(a_1,\ldots,a_m,a_{\overline{1}},\ldots,a_{\overline{n}}). \tag{55}$$

To complete the proof, it suffices to show that for all $\xi \in \mathfrak{a}^*$ we have

$$d_\lambda(\xi) = \overline{p}_\lambda\big(a_1,\ldots,a_m,a_{\overline{1}},\ldots,a_{\overline{n}}\big), \tag{56}$$

where $\overline{p}_\lambda$ denotes the homogeneous part of highest degree of p_λ. Set $\mathbf{D}_i := \check{\rho}(h_i) \in \mathscr{PD}(W)$ for every $i \in \mathcal{I}_{m,n}$. Since $\mathfrak{a}$ is commutative, we have $\mathbf{D}_i\mathbf{D}_j = \mathbf{D}_j\mathbf{D}_i$ for every $i,j \in \mathcal{I}_{m,n}$, and thus

$$\check{\rho}(z_{\lambda,\mathfrak{a}}) = p_\lambda(\mathbf{D}_1,\ldots,\mathbf{D}_m,\mathbf{D}_{\overline{1}},\ldots,\mathbf{D}_{\overline{n}}).$$

The last relation together with the fact that $\deg(p_\lambda) = d$ imply that

$$\widehat{\mathsf{s}}_d(\check{\rho}(z_{\lambda,\mathfrak{a}})) = \overline{p}_\lambda\big(\widehat{\mathsf{s}}_1(\mathbf{D}_1),\ldots,\widehat{\mathsf{s}}_m(\mathbf{D}_m),\widehat{\mathsf{s}}_1(\mathbf{D}_{\overline{1}}),\ldots,\widehat{\mathsf{s}}_1(\mathbf{D}_{\overline{n}})\big),$$

Using (53) we obtain

$$\widetilde{\mathsf{h}}_\beta(\widehat{\mathsf{s}}_d(\check{\rho}(z_{\lambda,\mathfrak{a}}))) = \overline{p}_\lambda\big(-y_{1,1},\ldots,-y_{m,m},-2y_{\overline{1},\overline{2}},\ldots,-2y_{\overline{2n-1},\overline{2n}}\big). \tag{57}$$

It is straightforward to verify that

$$\langle\iota_{\mathfrak{a}}^*(y_{k,k}),\mathsf{j}(\xi)\rangle = -a_k \ \text{ for } 1\leq k\leq m \ \text{ and } \ \langle\iota_{\mathfrak{a}}^*(y_{\overline{2l-1},\overline{2l}}),\mathsf{j}(\xi)\rangle = -\frac{1}{2}a_{\overline{l}} \ \text{ for } 1\leq l\leq n.$$

Thus, by composing both sides of (57) with $\mathsf{j}^*\circ\iota_{\mathfrak{a}}^*$ and then evaluating both sides at ξ, we obtain

$$d_\lambda(\xi) = \big(\mathsf{j}^*\circ\iota_{\mathfrak{a}}^*\circ\widetilde{\mathsf{h}}_\beta\circ\widehat{\mathsf{s}}_d\big)(\check{\rho}(z_{\lambda,\mathfrak{a}}))(\xi) = \overline{p}_\lambda\big(a_1,\ldots,a_m,a_{\overline{1}},\ldots,a_{\overline{n}}\big).$$

This establishes (56) and completes the proof. ∎

Our final result in this section is a characterization of the eigenvalue polynomial c_λ by its symmetry and vanishing properties. Let

$$\mathrm{HC}^+ : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{a}) \tag{58}$$

denote the Harish–Chandra projection defined by the composition

$$\mathbf{U}(\mathfrak{g}) \xrightarrow{\mathsf{p}^+} \mathbf{U}(\mathfrak{h}) \xrightarrow{\mathsf{q}} \mathbf{U}(\mathfrak{a})$$

where $\mathsf{p}^+ : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{h})$ is the projection according to the decomposition

$$\mathbf{U}(\mathfrak{g}) = \big(\mathbf{U}(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^-\mathbf{U}(\mathfrak{g})\big) \oplus \mathbf{U}(\mathfrak{h})$$

and $\mathsf{q} : \mathbf{U}(\mathfrak{h}) \to \mathbf{U}(\mathfrak{a})$ is the projection corresponding to the decomposition $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$. For every $\lambda \in \mathrm{E}^*_{m,n}$, let $z_\lambda \in \mathbf{Z}(\mathfrak{g})$ be defined as in (44). A precise description of the algebra $\mathsf{p}^+(\mathbf{Z}(\mathfrak{g})) \subseteq \mathbf{U}(\mathfrak{h})$ is known (see for example [22], [24], [11], [7], or [5, Sec. 2.2.3]), and implies that $\mathsf{p}^+(\mathbf{Z}(\mathfrak{g}))$ is equal to the subalgebra of $\mathbf{U}(\mathfrak{h})$ generated by the elements

$$G_d := \sum_{k=1}^{m} \left(E_{k,k} + \frac{m+1}{2} - n - k \right)^d + (-1)^{d-1} \sum_{l=1}^{2n} \left(E_{\bar{l},\bar{l}} + \frac{m+1}{2} + n - l \right)^d \tag{59}$$

for every $d \geq 1$. Let $\mathbf{I}(\mathfrak{a}^*) \subseteq \mathscr{P}(\mathfrak{a}^*)$ be the subalgebra that corresponds to $\mathsf{q}(\mathsf{p}^+(\mathbf{Z}(\mathfrak{g}))$ via the canonical isomorphism $\mathbf{U}(\mathfrak{a}) \cong \mathcal{S}(\mathfrak{a}) \cong \mathscr{P}(\mathfrak{a}^*)$.

Theorem 5.9. *Let $\lambda \in \mathrm{E}^*_{m,n,d}$. Then c_λ is the unique element of $\mathbf{I}(\mathfrak{a}^*)$ that satisfies $\deg(c_\lambda) \leq d$, $c_\lambda(\lambda) = d!$, and $c_\lambda(\mu) = 0$ for all other μ in $\bigcup_{d'=0}^{d} \mathrm{E}^*_{m,n,d'}$.*

Proof. By Remark 5.3 and Lemma 5.4, only the uniqueness statement requires proof.

Step 1. We prove that

$$\mathbf{I}(\mathfrak{a}^*) \cap \bigoplus_{d'=0}^{d} \mathscr{P}^{d'}(\mathfrak{a}^*) = \mathrm{Span}_{\mathbb{C}} \left\{ c_\lambda \,:\, \lambda \in \bigcup_{d'=0}^{d} \mathrm{E}^*_{m,n,d'} \right\}. \tag{60}$$

Fix $z \in \mathbf{Z}(\mathfrak{g})$ and let $\mu \in \mathrm{E}^*_{m,n}$. Let $v_\mu \in V_\mu$ be a highest weight vector and let $v^*_\circ \in V^*_\mu$ be a nonzero $\mathfrak{k}$-invariant vector. As shown in the proof of Lemma 5.1, we have $\langle v^*_\circ, v_\mu \rangle \neq 0$. Since $z \in \mathbf{Z}(\mathfrak{g})$, we have $z - \mathsf{p}^+(z) \in \mathbf{U}(\mathfrak{g})\mathfrak{n}^+ \cap \mathfrak{n}^-\mathbf{U}(\mathfrak{g})$, and therefore

$$\begin{aligned} \langle v^*_\circ, \check{\rho}(z) v_\mu \rangle &= \langle v^*_\circ, \check{\rho}(\mathsf{p}^+(z)) v_\mu \rangle \\ &= \langle v^*_\circ, \check{\rho}(\mathsf{q}(\mathsf{p}^+(z))) v_\mu \rangle = \mu(\mathsf{q}(\mathsf{p}^+(z))) \langle v^*_\circ, v_\mu \rangle = \mu(\mathrm{HC}^+(z)) \langle v^*_\circ, v_\mu \rangle. \end{aligned}$$

Since $\check{\rho}(z) \in \mathscr{PD}(W)^{\mathfrak{g}}$, we can write $\check{\rho}(z)$ as a linear combination of Capelli operators (see Definition 4.5), say $\check{\rho}(z) = \sum_k a_k D_{\lambda_k}$. It follows that the map $\mu \mapsto \mu(\mathrm{HC}^+(z))$ agrees with $\sum_k a_k c_{\lambda_k}(\mu)$ for $\mu \in \mathrm{E}^*_{m,n}$. Since $\mathrm{E}^*_{m,n}$ is Zariski dense in $\mathfrak{a}^*$, the latter two maps should agree for all $\mu \in \mathfrak{a}^*$ as well. This implies that

$$\mathbf{I}(\mathfrak{a}^*) \subseteq \mathrm{Span}_{\mathbb{C}}\{c_\lambda \,:\, \lambda \in \mathrm{E}^*_{m,n}\}.$$

Furthermore, from Theorem 5.8 and Proposition 5.6 it follows that the homogeneous part of highest degree of every nonzero element of

$$\mathrm{Span}_{\mathbb{C}}\left\{c_\lambda \,:\, \lambda \in \bigcup_{d'>d} \mathrm{E}^*_{m,n,d'}\right\}$$

has degree strictly bigger than d. Consequently, the left hand side of (60) is a subset of its right hand side. The reverse inclusion follows from the above argument by choosing $z := z_\lambda$ where z_λ is defined in (44).

Step 2. Set $N_{m,n,d} := \sum_{d'=0}^{d} \left|\mathrm{E}^*_{m,n,d'}\right|$. From Step 1 it follows that

$$\dim\left(\mathbf{I}(\mathfrak{a}^*) \cap \bigoplus_{d'=0}^{d} \mathscr{P}^{d'}(\mathfrak{a}^*)\right) \leq N_{m,n,d}.$$

Now consider the linear map

$$L_{m,n,d} : \mathbf{I}(\mathfrak{a}^*) \cap \bigoplus_{d'=0}^{d} \mathscr{P}^{d'}(\mathfrak{a}^*) \to \mathbb{C}^{N_{m,n,d}}\ ,\ p \mapsto \big(p(\mu)\big)_{\mu \in \bigcup_{d'=0}^{d} \mathrm{E}^*_{m,n,d'}}.$$

From Lemma 5.4 it follows that the vectors $L_{m,n,d}(c_\lambda)$, for $\lambda \in \bigcup_{d'=0}^{d} \mathrm{E}^*_{m,n,d'}$, form a nonsingular triangular matrix, so that they form a basis of $\mathbb{C}^{N_{m,n,d}}$. Consequently, $L_{m,n,d}$ is an invertible linear transformation. In particular, for every $\lambda \in \mathrm{E}^*_{m,n,d}$, the polynomial c_λ is the unique element of $\mathbf{I}(\mathfrak{a}^*) \cap \bigoplus_{d'=0}^{d} \mathscr{P}^{d'}(\mathfrak{a}^*)$ that satisfies $c_\lambda(\lambda) = d!$ and $c_\lambda(\mu) = 0$ for all other $\mu \in \bigcup_{d'=0}^{d} \mathrm{E}^*_{m,n,d'}$. $\square$

6 Relation with Sergeev–Veselov polynomials for $\theta = \frac{1}{2}$

The main result of this section is Theorem 6.5, which describes the precise relation between the eigenvalue polynomials c_λ of Definition 5.2 and the shifted super Jack polynomials of Sergeev–Veselov [25]. As in the previous section, we set $V := \mathbb{C}^{m|2n}$, so that $\mathfrak{g} := \mathfrak{gl}(V) = \mathfrak{gl}(m|2n)$. The map $\omega : \mathfrak{g} \to \mathfrak{g}$ given by $\omega(x) = -x$ is an anti-automorphism of $\mathfrak{g}$, that is,

$$\omega([x,y]) = (-1)^{|x|\cdot|y|}[\omega(y), \omega(x)]$$

for $x, y \in \mathfrak{g}$. Therefore ω extends canonically to an anti-automorphism $\omega : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g})$, that is,

$$\omega(xy) = (-1)^{|x|\cdot|y|}\omega(y)\omega(x) \ \text{ for } x, y \in \mathbf{U}(\mathfrak{g}).$$

Recall that $\mathrm{HC}^+ : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{a})$ is the Harish–Chandra projection as in (58). Let

$$\mathrm{HC}^- : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{a})$$

be the opposite Harish–Chandra projection defined by the composition

$$\mathbf{U}(\mathfrak{g}) \xrightarrow{\mathsf{p}^-} \mathbf{U}(\mathfrak{h}) \xrightarrow{\mathsf{q}} \mathbf{U}(\mathfrak{a})$$

where where $\mathsf{p}^- : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{h})$ is the projection according to the decomposition

$$\mathbf{U}(\mathfrak{g}) = \big(\mathbf{U}(\mathfrak{g})\mathfrak{n}^- + \mathfrak{n}^+\mathbf{U}(\mathfrak{g})\big) \oplus \mathbf{U}(\mathfrak{h}).$$

It is straightforward to verify that

$$\mathrm{HC}^-(\omega(z)) = \omega(\mathrm{HC}^+(z)) \text{ for } z \in \mathbf{U}(\mathfrak{g}). \tag{61}$$

Definition 6.1. For every $\mu \in \mathrm{E}^*_{m,n}$, we use μ^* to denote the unique element of $\mathrm{E}_{m,n}$ that satisfies $V_\mu^* \cong V_{\mu^*}$.

Using formulas (32) and (35), one can see that the map $\mu \mapsto \mu^*$ is not linear. Nevertheless, the following proposition still holds.

Proposition 6.2. *Let $\lambda \in \mathrm{E}^*_{m,n,d}$. Then there exists a unique polynomial $c^*_\lambda \in \mathscr{P}(\mathfrak{a}^*)$ such that* $\deg(c^*_\lambda) \leq d$ *and $c_\lambda(\mu) = c^*_\lambda(\mu^*)$ for every $\mu \in \mathrm{E}^*_{m,n}$.*

Proof. First we prove the existence of c^*_λ. Recall that the action of $D_\lambda : V_\mu \to V_\mu$ is by the scalar $c_\lambda(\mu)$. Let v_{μ^*} denote the highest weight of V_{μ^*}, and define $v^-_\mu \in (V_{\mu^*})^* \cong V_\mu$ by $\langle v^-_\mu, v^*_\mu\rangle = 1$ and $\langle v^-_\mu, \mathbf{U}(\mathfrak{n}^-)v_{\mu^*}\rangle = 0$. It is straightforward to verify that $v^-_\mu \in (V_\mu)^{\mathfrak{n}^-}$, and thus v^-_μ is the lowest weight vector of V_μ. It follows immediately that the lowest weight of V_μ is $-\mu^*$. Choose $z_\lambda \in \mathbf{Z}(\mathfrak{g})$ as in (44). By considering the $\mathfrak{h}$-action on $\mathbf{U}(\mathfrak{g})$ we obtain $z_\lambda - \mathsf{p}^-(z_\lambda) \in \mathbf{U}(\mathfrak{g})\mathfrak{n}^-$, and therefore

$$c_\lambda(\mu)v^-_\mu = D_\lambda v^-_\mu = \check{\rho}(z_\lambda)v^-_\mu = \check{\rho}(\mathsf{p}^-(z_\lambda))v^-_\mu = \mathsf{h}_{-\mu^*}(\mathrm{HC}^-(z_\lambda))v^-_\mu,$$

where $\mathsf{h}_{-\mu^*} : \mathbf{U}(\mathfrak{a}) \cong \mathcal{S}(\mathfrak{a}) \to \mathbb{C}$ is defined as in (6). It is straightforward to check that the map

$$c^*_\lambda(\nu) := \mathsf{h}_{-\nu}(\mathrm{HC}^-(z_\lambda)) \tag{62}$$

is a polynomial in $\nu \in \mathfrak{a}^*$. From (61) and Theorem 3.9 it follows that $\deg(c^*_\lambda) \leq d$. Finally, uniqueness of c^*_λ follows from the fact that $\mathrm{E}_{m,n}$ is Zariski dense in $\mathfrak{a}^*$. ☐

We now recall the definition of the algebra of *shifted supersymmetric polynomials*

$$\Lambda^\natural_{m,n,\frac{1}{2}} \subseteq \mathscr{P}(\mathbb{C}^{m+n}),$$

introduced in [25, Sec. 6]. For $1 \leq k \leq r$, let $\mathsf{e}_{k,r}$ be the k-th unit vector in $\mathbb{C}^r$. Then the algebra $\Lambda^\natural_{m,n,\frac{1}{2}}$ consists of polynomials $f(\mathsf{x}_1, \ldots, \mathsf{x}_m, \mathsf{y}_1, \ldots, \mathsf{y}_n)$, which are separately symmetric in $\mathsf{x} := (\mathsf{x}_1, \ldots, \mathsf{x}_m)$ and in $\mathsf{y} := (\mathsf{y}_1, \ldots, \mathsf{y}_n)$, and which satisfy the relation

$$f(\mathsf{x} + \tfrac{1}{2}\mathsf{e}_{k,m}, \mathsf{y} - \tfrac{1}{2}\mathsf{e}_{l,n}) = f(\mathsf{x} - \tfrac{1}{2}\mathsf{e}_{k,m}, \mathsf{y} + \tfrac{1}{2}\mathsf{e}_{l,n})$$

on every hyperplane $\mathsf{x}_k + \frac{1}{2}\mathsf{y}_l = 0$, where $1 \leq k \leq m$ and $1 \leq l \leq n$.

In [25, Sec. 6], Sergeev and Veselov introduce a basis of $\Lambda^\natural_{m,n,\frac{1}{2}}$,

$$\left\{ SP^*_\flat \in \Lambda^\natural_{m,n,\frac{1}{2}} \ : \ \flat \in \mathrm{H}_{m,n} \right\}, \tag{63}$$

indexed by the set $\mathrm{H}_{m,n}$ of $(m|n)$-hook partitions (see Definition 4.3). The polynomials $SP^*_\flat$ are called *shifted super Jack polynomials* and they satisfy certain vanishing conditions, given in [25, Eq. (31)].

Recall the map $\mathbf{\Gamma} : \mathrm{H}_{m,n} \to \mathfrak{a}^*$ from (31). Given $\flat = (\flat_1, \flat_2, \flat_3, \dots) \in \mathrm{H}_{m,n}$, we set $\flat^*_k := \max\{\flat'_k - m, 0\}$ for every $k \geq 1$. Fix $\mu \in \mathrm{E}^*_{m,n}$, and let $\mu^* \in \mathrm{E}_{m,n}$ be as in Definition 6.1. From the definition of the map $\mathbf{\Gamma}$ it follows that there exists a $\flat \in \mathrm{H}_{m,n}$ such that

$$\mu^* = \mathbf{\Gamma}(\flat) = \sum_{i=1}^{m} 2\flat_i \gamma_i + \sum_{j=1}^{n} 2\flat^*_j \gamma_{\overline{j}}. \tag{64}$$

Now let $\mathsf{F} : \mathrm{H}_{m,n} \to \mathbb{C}^{m+n}$ be the *Frobenius map* of [25, Sec. 6], defined as follows. For every $\flat \in \mathrm{H}_{m,n}$, we have $\mathsf{F}(\flat) = (\mathsf{x}_1(\flat), \dots, \mathsf{x}_m(\flat), \mathsf{y}_1(\flat), \dots, \mathsf{y}_n(\flat))$, where

$$\begin{cases} \mathsf{x}_k(\flat) := \flat_k - \frac{1}{2}(k - \frac{1}{2}) - \frac{1}{2}(2n - \frac{m}{2}) & \text{for } 1 \leq k \leq m, \\ \mathsf{y}_l(\flat) := \flat^*_l - 2(l - \frac{1}{2}) + \frac{1}{2}(4n + m) & \text{for } 1 \leq l \leq n. \end{cases} \tag{65}$$

Lemma 6.3. *The map $\mathbf{\Gamma} \circ \mathsf{F}^{-1}$ defined initially on $\mathsf{F}(\mathrm{H}_{m,n})$ extends uniquely to an affine linear map $\Psi : \mathbb{C}^{m+n} \to \mathfrak{a}^*$.*

Proof. This is immediate from the formulas (64) and (65), and the Zariski density of $\mathsf{F}(\mathrm{H}_{m,n})$ in $\mathbb{C}^{m+n}$. $\square$

Definition 6.4. For $p \in \mathscr{P}(\mathfrak{a}^*)$, we define its *Frobenius transform* to be

$$\mathscr{F}(p) := p \circ \Psi.$$

Note that

$$\deg(p) = \deg(\mathscr{F}(p)) \text{ for every } p \in \mathscr{P}(\mathfrak{a}^*). \tag{66}$$

We now relate the shifted super Jack polynomials $SP^*_\flat$ to the dualized eigenvalue polynomials c^*_λ from Proposition 6.2.

Theorem 6.5. *Let $\lambda \in \mathrm{E}^*_{m,n,d}$ and let $\flat \in \mathrm{H}_{m,n,d}$ be such that $\lambda^* = \mathbf{\Gamma}(\flat)$. Then we have*

$$\mathscr{F}(c^*_\lambda) = \frac{d!}{H(\flat)} SP^*_\flat, \tag{67}$$

where $H(\flat) := \prod_{k \geq 1} \prod_{l=1}^{\flat_k} \left(\flat_k - l + 1 + \frac{1}{2}(\flat'_l - k)\right)$.

Proof. By (66), Proposition 6.2, and the results of [25, Sec. 6], both sides of (67) are in $\bigoplus_{d'=0}^{d} \mathscr{P}^{d'}(\mathbb{C}^{m+n})$. Next we prove that

$$\mathscr{F}(c^*_\lambda) \in \Lambda^\natural_{m,n,\frac{1}{2}}. \tag{68}$$

Note that $\mathbf{Z}(\mathfrak{g})$ is invariant under the anti-automorphism ω. Let $z_\lambda \in \mathbf{Z}(\mathfrak{g}) \cap \mathbf{U}^d(\mathfrak{g})$ be as in (44). From (61) we obtain $\mathrm{HC}^-(z_\lambda) = \omega(\mathrm{HC}^+(\omega(z_\lambda)))$, and thus $\mathrm{HC}^-(z_\lambda)$ is in the subalgebra of $\mathbf{U}(\mathfrak{a}) \cong \mathcal{S}(\mathfrak{a})$ that is generated by $\omega(\mathfrak{q}(G_d))$ for $d \geq 1$, where G_d is given in (59). Observe that $\mathfrak{q}(G_d)$ is obtained from G_d by the substitutions

$$E_{k,k} \mapsto h_k \text{ for } 1 \leq k \leq m \quad \text{and} \quad E_{\overline{2l-1},\overline{2l-1}}, E_{\overline{2l},\overline{2l}} \mapsto \frac{1}{2} h_{\overline{l}} \text{ for } 1 \leq l \leq n,$$

where $\{h_i : i \in \mathcal{I}_{m,n}\}$ is the basis for $\mathfrak{a}$ that is dual to the $\mathfrak{a}^*$-basis $\{\gamma_i : i \in \mathcal{I}_{m,n}\}$, defined in (29). By a straightforward calculation we can verify that $\mathscr{F}(c^*_\lambda)$ is in the algebra generated by the polynomials

$$\sum_{k=1}^{m}(2\mathsf{x}_k + n)^d + (-1)^{d-1}\sum_{l=1}^{n}\left(\mathsf{y}_l - n + \frac{1}{2}\right)^d + \left(\mathsf{y}_l - n - \frac{1}{2}\right)^d \tag{69}$$

for every $d \geq 1$. Furthermore, the polynomials (69) belong to $\Lambda^{\natural}_{m,n,\frac{1}{2}}$. This completes the proof of (68).

Next we fix $\mu \in \mathrm{E}^*_{m,n}$ and choose $\flat^{(\mu)} \in \mathrm{H}_{m,n}$ such that $\mu^* = \boldsymbol{\Gamma}(\flat^{(\mu)})$, where $\mu^* \in \mathrm{E}_{m,n}$ is as in Definition 6.1. Then

$$\mathscr{F}(c^*_\lambda)\left(\mathsf{F}\left(\flat^{(\mu)}\right)\right) = c^*_\lambda(\mu^*) = c_\lambda(\mu).$$

From [25, Eq. (31)] we have $SP^*_\flat\left(\mathsf{F}\left(\flat^{(\mu)}\right)\right) = H\left(\flat^{(\mu)}\right)$ for $\mu = \lambda$, and $SP^*_\flat\left(\mathsf{F}\left(\flat^{(\mu)}\right)\right) = 0$ for all other $\mu \in \bigcup_{d'=0}^{d} \mathrm{E}^*_{m,n,d'}$. The equality (67) now follows from the latter vanishing property and Theorem 5.9, or alternatively from the discussion immediately above [25, Eq. (31)]. $\square$

Appendix

A Proof of Proposition 5.6

In this appendix we prove Proposition 5.6. The proof of this proposition is similar to the proof of Proposition 4.6, although somewhat more elaborate. Recall the generators $\{x_{i,j}\}_{i,j\in\mathcal{I}_{m,2n}}$ for $\mathcal{S}(W)$ and $\{y_{i,j}\}_{i,j\in\mathcal{I}_{m,2n}}$ for $\mathscr{P}(W)$, defined in (15). We consider the total ordering $\prec$ on $\mathcal{I}_{m,2n}$ given by

$$1 \prec \cdots \prec m \prec \overline{1} \prec \cdots \prec \overline{2n}.$$

Set $\mathcal{I}' := \left\{\left(\overline{2k-1}, \overline{2k}\right) : 1 \leq k \leq n\right\}$ and $\mathcal{I}'' := \{(i,j) \in \mathcal{I}_{m,2n} \times \mathcal{I}_{m,2n}\ ,\ i \prec j\}\backslash\mathcal{I}'$. For every $a_1, \ldots, a_m, a_{\overline{1}}, \ldots, a_{\overline{n}} \in \mathbb{C}$, we set

$$\mathbf{x} := -\sum_{k=1}^{m} a_k h_k + \sum_{l=1}^{n} a_{\overline{l}} h_{\overline{l}} \in \mathfrak{a}, \tag{70}$$

where $\{h_i : i \in \mathcal{I}_{m,n}\}$ is the basis of $\mathfrak{a}$ that is dual to $\{\gamma_i : i \in \mathcal{I}_{m,n}\}$ defined in (29). Let $\mathsf{h}_\mathbf{x} : \mathscr{P}(\mathfrak{a}) \cong \mathcal{S}(\mathfrak{a}^*) \to \mathbb{C}$ be as defined in (6). For any $\mathbf{x} \in \mathfrak{a}$ as above, set

$$\xi_\mathbf{x} := \frac{1}{2}\sum_{k=1}^{m} a_k x_{k,k} + \sum_{l=1}^{n} a_{\overline{l}} x_{\overline{2l-1},\overline{2l}} \in W,$$

and let $\mathsf{h}_{\xi_\mathbf{x}} : \mathscr{P}(W) \cong \mathcal{S}(W^*) \to \mathbb{C}$ be as defined in (6). Observe that $\mathsf{h}_\mathbf{x} \circ \iota^*_\mathfrak{a} = \mathsf{h}_{\xi_\mathbf{x}}$ where $\iota^*_\mathfrak{a}$ is as defined in (48). In particular, for every $a \in \mathscr{P}(W)$, the set

$$S_a := \{\mathbf{x} \in \mathfrak{a}\ :\ \mathsf{h}_{\xi_\mathbf{x}}(a) = 0\}$$

is Zariski closed in $\mathfrak{a}$.

Let $\partial_{i,j} := \partial_{x_{i,j}}$ denote the superderivation of $\mathscr{P}(W)$ that is defined according to (8).

Proposition A.1. *Let* $\mathbf{d} \in \mathscr{P}(W)^{\mathfrak{k}}$ *such that* $\mathsf{h}_{\xi_{\mathbf{x}}}(\mathbf{d}) = 0$ *for every* $\mathbf{x} \in \mathfrak{a}$. *Then*

$$\mathsf{h}_{\xi_{\mathbf{x}}}\big(\partial_{i_1,j_1} \cdots \partial_{i_s,j_s}\mathbf{d}\big) = 0 \quad \textit{for all } \mathbf{x} \in \mathfrak{a},\ s \geq 1,\ \textit{and } (i_1,j_1),\ldots,(i_s,j_s) \in \mathcal{I}''. \tag{71}$$

Proof. We use induction on s. First assume that $s = 1$. Then

$$\check{\rho}(x)\mathbf{d} \text{ for all } x \in \mathfrak{k}. \tag{72}$$

Set $x := E_{k,l} - E_{l,k}$ for $1 \leq k < l \leq m$. Then (72) and (18) imply that

$$\begin{aligned} 0 &= \mathsf{h}_{\xi_{\mathbf{x}}}(\check{\rho}(x)\mathbf{d}) \\ &= -\sum_{r \in \mathcal{I}_{m,2n}} (-1)^{|r|}\mathsf{h}_{\xi_{\mathbf{x}}}(y_{r,l})\mathsf{h}_{\xi_{\mathbf{x}}}(\partial_{r,k}\mathbf{d}) + \sum_{r \in \mathcal{I}_{m,2n}} (-1)^{|r|}\mathsf{h}_{\xi_{\mathbf{x}}}(y_{r,k})\mathsf{h}_{\xi_{\mathbf{x}}}(\partial_{r,l}\mathbf{d}) \\ &= (-a_l + a_k)\mathsf{h}_{\xi_{\mathbf{x}}}(\partial_{k,l}\mathbf{d}), \end{aligned}$$

from which it follows that $\mathsf{h}_{\xi_{\mathbf{x}}}(\partial_{k,l}d_\lambda) = 0$ for all $\mathbf{x} \in \mathfrak{a}$ as in (70) which satisfy $a_k \neq a_l$. But the set of all $\mathbf{x} \in \mathfrak{a}$ given as in (70) which satisfy $a_k \neq a_l$ for all $1 \leq k < l \leq m$ is a Zariski dense subset of $\mathfrak{a}$, and it follows that $\mathsf{h}_{\xi_{\mathbf{x}}}(\partial_{x_{k,l}}\mathbf{d}) = 0$ for every $\mathbf{x} \in \mathfrak{a}$. A similar argument for each of the cases (ii)–(vi) of Remark 4.1 (where in cases (ii)–(iv) we assume $k \neq l$) proves (71) for $s = 1$.

Next we define an involution $i \mapsto i^\dagger$ on $\mathcal{I}_{m,2n}$ by

$$k^\dagger := k \text{ for } 1 \leq k \leq m,\ (\overline{2l-1})^\dagger := \overline{2l} \text{ for } 1 \leq l \leq n, \text{ and } (\overline{2l})^\dagger := \overline{2l-1} \text{ for } 1 \leq l \leq n.$$

Let $\mathsf{f} : \mathcal{I}_{m,2n} \to \mathcal{I}_{m,n}$ be defined by $\mathsf{f}(k) = k$ for $1 \leq k \leq m$ and $\mathsf{f}\big(\overline{2l-1}\big) = \mathsf{f}\big(\overline{2l}\big) = \bar{l}$ for $1 \leq l \leq n$. Fix an element $x \in \mathfrak{k}$ that belongs to the spanning set of $\mathfrak{k}$ given in Remark 4.1. (When x is chosen from one of the cases (ii)–(iv) in Remark 4.1, we assume that $k \neq l$.) Then there exist $p, q \in \mathcal{I}_{m,2n}$ such that

$$p \prec q,\ p^\dagger \prec q^\dagger, \text{ and } \check{\rho}(x) = \sum_{r \in \mathcal{I}_{m,2n}} \Big(\pm y_{r,p}\partial_{r,q} \pm y_{r,q^\dagger}\partial_{r,p^\dagger}\Big). \tag{73}$$

Now fix $x_1, \ldots, x_{s+1} \in \mathfrak{k}$ such that every x_k, for $1 \leq k \leq s+1$, is an element of the spanning set of $\mathfrak{k}$ given in Remark 4.1. For every $1 \leq k \leq s+1$, if x_k is chosen from the cases (ii)–(iv) in Remark 4.1, then we assume that $k \neq l$. Choose $(p_1, q_1), \ldots, (p_{s+1}, q_{s+1}) \in \mathcal{I}_{m,2n} \times \mathcal{I}_{m,2n}$ corresponding to $x_1, \ldots, x_{s+1}$ which satisfy (73). For $1 \leq u \leq s+1$, we define

$$p_{u,S} := \begin{cases} p_u & \text{if } u \in S, \\ (q_u)^\dagger & \text{if } u \notin S, \end{cases} \quad \text{and} \quad q_{u,S} := \begin{cases} q_u & \text{if } u \in S, \\ (p_u)^\dagger & \text{if } u \notin S. \end{cases}$$

Then

$$\begin{aligned} &\check{\rho}(x_1) \cdots \check{\rho}(x_{s+1})\mathbf{d} \\ &= \sum_{S \subseteq \{1,\ldots,s+1\}} \sum_{r_1,\ldots,r_{s+1} \in \mathcal{I}_{m,2n}} \Big(\pm y_{r_1,p_{1,S}} \cdots y_{r_{s+1},p_{s+1,S}}\partial_{r_1,q_{1,S}} \cdots \partial_{r_{s+1},q_{s+1,S}}(\mathbf{d})\Big) + R_{\mathbf{d}}, \end{aligned} \tag{74}$$

where $R_{\mathbf{d}}$ is a sum of terms of the form $b\partial_{p'_1,q'_1}\cdots\partial_{p'_t,q'_t}(\mathbf{d})$, with $b \in \mathscr{P}(W)$ and $t \leq s$. By the induction hypothesis, $\mathsf{h}_{\xi_\mathbf{x}}(R_{\mathbf{d}}) = 0$ for every $\mathbf{x} \in \mathfrak{a}$. Thus (74) implies that

$$\begin{aligned} 0 &= \mathsf{h}_{\xi_\mathbf{x}}(\check{\rho}(x_1)\cdots\check{\rho}(x_{s+1})\mathbf{d}) \\ &= \sum_{S\subseteq\{1,\dots,s+1\}} \pm a_{\mathsf{f}(p_{1,S})}\cdots a_{\mathsf{f}(p_{s+1,S})}\mathsf{h}_{\xi_\mathbf{x}}\left(\partial_{(p_1)^\dagger,q_1}\cdots\partial_{(p_{s+1})^\dagger,q_{s+1}}\mathbf{d}\right). \end{aligned} \tag{75}$$

From (73) it follows that $p_u \prec (q_u)^\dagger$ for every $1 \leq u \leq s+1$. the monomial

$$a_{\mathsf{f}(p_1)}\cdots a_{\mathsf{f}(p_{s+1})}$$

appears in (75) exactly once. Therefore the right hand side of (75) can be expressed as

$$\psi(\mathbf{x})\mathsf{h}_{\xi_\mathbf{x}}\left(\partial_{(p_1)^\dagger,q_1}\cdots\partial_{(p_{s+1})^\dagger,q_{s+1}}\mathbf{d}\right),$$

where $\psi \in \mathscr{P}(\mathfrak{a})$ is a nonzero polynomial. It follows that the set consisting of all $\mathbf{x} \in \mathfrak{a}$ which satisfy $\psi(\mathbf{x}) \neq 0$ is a Zariski dense subset of $\mathfrak{a}$. Consequently, the set

$$\left\{\mathbf{x} \in \mathbb{C}^{m+n} \ : \ \mathsf{h}_{\xi_\mathbf{x}}\left(\partial_{(p_1)^\dagger,q_1}\cdots\partial_{(p_{s+1})^\dagger,q_{s+1}}\mathbf{d}\right) = 0\right\}$$

is both Zariski dense and Zariski closed. This completes the proof of (71). $\square$

We are now ready to prove Proposition 5.6.

Proof. By the equality $\mathsf{h}_\mathbf{x} \circ \iota^*_\mathfrak{a} = \mathsf{h}_{\xi_\mathbf{x}}$, it suffices to show that for every $\mathbf{d} \in \mathscr{P}(W)^\mathfrak{k}$, if $\mathsf{h}_{\xi_\mathbf{x}}(\mathbf{d}) = 0$ for every $\mathbf{x} \in \mathfrak{a}$, then $\mathbf{d} = 0$. Let $\mathscr{D} \subseteq \mathscr{P}(W)$ be the subalgebra generated by

$$\left\{y_{k,k} \ : \ 1 \leq k \leq m\right\} \cup \left\{y_{\overline{2l-1,2l}} \ : \ 1 \leq l \leq n\right\}.$$

Then $\mathbf{d} = \sum_{S\subseteq\mathcal{I}''} a_S y_S$, where $y_S := \prod_{(i,j)\in S} y_{i,j}$ and $a_S \in \mathscr{D}$ for every $S \subseteq \mathcal{I}''$. Now we fix $S \subseteq \mathcal{I}''$ and set $\widetilde{\partial} := \prod_{(i,j)\in S}\partial_{i,j}$, so that $\widetilde{\partial}(\mathbf{d}) = za_S$ for some scalar $z \neq 0$. By (71),

$$\mathsf{h}_{\xi_\mathbf{x}}(a_S) = \frac{1}{z}\mathsf{h}_{\xi_\mathbf{x}}\left(\widetilde{\partial}(\mathbf{d})\right) = 0 \text{ for every } \mathbf{x} \in \mathfrak{a}. \tag{76}$$

From (76) it follows that $a_S = 0$. Since $S \subseteq \mathcal{I}''$ is arbitrary, we obtain $\mathbf{d} = 0$. $\square$

B The Capelli problem for $\mathfrak{gl}(V) \times \mathfrak{gl}(V)$ acting on $V \otimes V^*$

In this appendix, we show that the main results of Sections 3–6, including the abstract Capelli theorem and the relation between the eigenvalue polynomials c_λ and the shifted super Jack polynomials of [25, Eq. (31)], extend to the case of $\mathfrak{gl}(V) \times \mathfrak{gl}(V)$ acting on $W := V \otimes V^*$, where $V := \mathbb{C}^{m|n}$. These extensions can be proved along the same lines. However, they can also be deduced from the results of [16], and we sketch the necessary arguments below.

Recall the triangular decomposition $\mathfrak{gl}(m|n) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ from Section 3. Set

$$\mathfrak{g} := \mathfrak{gl}(V) \times \mathfrak{gl}(V) \cong \mathfrak{gl}(m|n) \times \mathfrak{gl}(m|n). \tag{77}$$

Recall that $\mathrm{H}_{m,n}$ is the set of (m,n)-hook partitions, as in Definition 4.3. We define a map $\mathbf{\Gamma} : \mathrm{H}_{m,n} \to \mathfrak{h}^*$ by $\mathbf{\Gamma}(\flat) := \sum_{k=1}^{m} \flat_k \varepsilon_k + \sum_{l=1}^{n} \flat_l^* \varepsilon_{\bar{l}}$, where $\flat_l^* := \max\{\flat_l' - m, 0\}$. Set $\breve{\mathrm{E}}_{m,n,d} := \mathbf{\Gamma}(\mathrm{H}_{m,n,d})$, and let $\breve{\mathrm{E}}_{m,n} := \bigcup_{d=0}^{\infty} \breve{\mathrm{E}}_{m,n,d}$. For every $\flat \in \mathrm{H}_{m,n}$, let $s_\flat$ denote the *supersymmetric Schur polynomial* defined in [16, Eq. (0.2)], and let $s_\flat^*$ denote the *shifted supersymmetric Schur polynomial* defined in [16, Sec. 7].

We can decompose $\mathscr{P}(W)$ and $\mathcal{S}(W)$ into irreducible $\mathfrak{g}$-modules, that is,

$$\mathscr{P}^d(W) \cong \bigoplus_{\mu \in \breve{\mathrm{E}}_{m,n,d}} V_\mu^* \otimes V_\mu \text{ and } \mathcal{S}^d(W) \cong \bigoplus_{\mu \in \breve{\mathrm{E}}_{m,n,d}} V_\mu \otimes V_\mu^*, \tag{78}$$

where V_μ is the irreducible $\mathfrak{gl}(V)$-module of highest weight μ, and V_μ^* is the contragredient of V_μ. An argument similar to the proof of Lemma 3.7, and based on a decomposition similar to (38), implies that

$$\mathscr{P}\mathscr{D}^d(W)^{\mathfrak{g}} \cong \bigoplus_{k=0}^{d} \left(\mathscr{P}^k(W) \otimes \mathcal{S}^k(W)\right)^{\mathfrak{g}} \cong \bigoplus_{k=0}^{d} \bigoplus_{\lambda \in \breve{\mathrm{E}}_{m,n,k}} \mathbb{C} D_\lambda,$$

where D_λ is the $\mathfrak{g}$-invariant differential operator that corresponds to the identity map in $\mathrm{Hom}_{\mathfrak{gl}(V)\times\mathfrak{gl}(V)}(V_\lambda^* \otimes V_\lambda, V_\lambda^* \otimes V_\lambda)$. As in (18), the $\mathfrak{g}$-action on $\mathscr{P}(W)$ can be realized by polarization operators, and therefore we obtain a Lie superalgebra homomorphism $\mathfrak{g} \to \mathscr{P}\mathscr{D}(W)$. The latter map extends to a homomorphism of associative superalgebras

$$\check{\rho} : \mathbf{U}(\mathfrak{g}) \to \mathscr{P}\mathscr{D}(W). \tag{79}$$

There exists a canonical tensor product decomposition $\mathbf{U}(\mathfrak{g}) \cong \mathbf{U}(\mathfrak{gl}(V)) \otimes \mathbf{U}(\mathfrak{gl}(V))$ corresponding to (77). Set $\mathbf{Z}^d(\mathfrak{gl}(V)) := \mathbf{Z}(\mathfrak{gl}(V)) \cap \mathbf{U}^d(\mathfrak{gl}(V))$ for every integer $d \geq 0$. The next theorem extends Theorem 3.9.

Theorem B.1. (Abstract Capelli Theorem for $W := V \otimes V^*$.) *The restrictions*

$$\mathbf{Z}^d(\mathfrak{gl}(V)) \otimes 1 \to \mathscr{P}\mathscr{D}^d(W)^{\mathfrak{g}} \text{ and } 1 \otimes \mathbf{Z}^d(\mathfrak{gl}(V)) \to \mathscr{P}\mathscr{D}^d(W)^{\mathfrak{g}}$$

of the map $\check{\rho}$ given in (79) *are surjective.*

Proof. The statement is a consequence of the results of [16]. In [16, Thm 7.5], a family of elements $\mathbb{S}_\flat \in \mathbf{Z}^{|\flat|}(\mathfrak{gl}(V))$ is constructed which is parametrized by partitions $\flat \in \mathrm{H}_{m,n}$. Furthermore, in [16, Thm 8.1] it is proved that the map $\mathbf{Z}^{|\flat|}(\mathfrak{gl}(V)) \otimes 1 \to \mathscr{P}\mathscr{D}^{|\flat|}(W)^{\mathfrak{g}}$ takes $\mathbb{S}_\flat$ to a differential operator $\Delta_\flat \in \mathscr{P}\mathscr{D}^{|\flat|}(W)^{\mathfrak{g}}$, which is defined in [16, Sec. 8]. To complete the proof of surjectivity of the map $\mathbf{Z}^d(\mathfrak{gl}(V)) \otimes 1 \to \mathscr{P}\mathscr{D}^d(W)^{\mathfrak{g}}$, it is enough to show that for every $d \geq 0$, the sets

$$\left\{\Delta_\flat \,:\, \flat \in \mathrm{H}_{m,n,k}, 0 \leq k \leq d\right\} \text{ and } \left\{D_\lambda \,:\, \lambda \in \breve{\mathrm{E}}_{m,n,k}, 0 \leq k \leq d\right\}$$

span the same subspace of $\mathscr{P}\mathscr{D}^d(W)^{\mathfrak{g}}$. Since the above two sets have an equal number of elements, it is enough to show that the elements of $\left\{\Delta_\flat \,:\, \flat \in \mathrm{H}_{m,n,k}, 0 \leq k \leq d\right\}$ are linearly independent. To prove the latter statement, we note that the spectrum of $\Delta_\flat$ can be expressed in terms of the Harish-Chandra image of $\mathbb{S}_\flat$, which by [16, Thm 7.5] is equal to the shifted supersymmetric Schur polynomial $s_\flat^*$. Since the $s_\flat^*$ are linearly independent (see [16, Cor. 7.2] and subsequent remarks therein), the operators $\Delta_\flat$ are also linearly independent. $\square$

For $\mu \in \breve{\mathrm{E}}_{m,n}$, set $W_\mu := V_\mu^* \otimes V_\mu$. The operator D_λ acts on W_μ by a scalar $c_\lambda(\mu) \in \mathbb{C}$. For $\flat \in \mathrm{H}_{m,n}$, let $\breve{H}(\flat)$ denote the product of the hook lengths of all of the boxes in the Young diagram representation of $\flat$. We define a map

$$\breve{\mathbf{\Gamma}}_\circ : \breve{\mathrm{E}}_{m,n} \to \mathbb{C}^{m+n} \ , \ \sum_{k=1}^{m} \flat_k \varepsilon_k + \sum_{l=1}^{n} \flat_l^* \varepsilon_{\bar{l}} \mapsto (\flat_1, \ldots, \flat_m, \flat_1^*, \ldots, \flat_n^*).$$

As in Section 5, we denote the homogeneous part of highest degree of any $p \in \mathscr{P}(\mathfrak{h}^*)$ by $\overline{p}$.

Theorem B.2. *Let $\lambda \in \breve{\mathrm{E}}_{m,n,d}$ and let $\flat \in \mathrm{H}_{m,n,d}$ be such that $\mathbf{\Gamma}(\flat) = \lambda$. Then*

$$c_\lambda = \frac{d!}{\breve{H}(\flat)} s_\flat^* \circ \breve{\mathbf{\Gamma}}_\circ.$$

In particular, $\overline{c}_\lambda = \frac{d!}{\breve{H}(\flat)} s_\flat \circ \breve{\mathbf{\Gamma}}_\circ$.

Proof. Follows from [16, Thm 7.3] and [16, Thm 7.5]. □

The Frobenius map

$$\mathsf{F} : \mathrm{H}_{m,n} \to \mathbb{C}^{m+n}$$

of [25, Sec. 6] is given by $\mathsf{F}(\flat) := (\mathsf{x}(\flat), \ldots, \mathsf{x}_m(\flat), \mathsf{y}_1(\flat), \cdots, \mathsf{y}_n(\flat))$, where

$$\begin{cases} \mathsf{x}_k(\flat) := \flat_k - k + \frac{1}{2}(1 - n + m) & \text{for } 1 \le k \le m, \\ \mathsf{y}_l(\flat) := \flat_l^* - l + \frac{1}{2}(m + n + 1) & \text{for } 1 \le l \le n. \end{cases}$$

As in Lemma 6.3, the map $\mathbf{\Gamma} \circ \mathsf{F}^{-1}$ extends to an affine linear map $\Psi : \mathbb{C}^{m+n} \to \mathfrak{h}^*$. Thus we can define the Frobenius transform

$$\mathscr{F} : \mathscr{P}(\mathfrak{h}^*) \to \mathscr{P}(\mathbb{C}^{m+n})$$

as in Definition 6.4, namely by $\mathscr{F}(p) := p \circ \Psi$.

For $\lambda \in \breve{\mathrm{E}}_{m,n,d}$, by Theorem B.1 there exists an element $z_\lambda \in \mathbf{Z}(\mathfrak{gl}(V)) \cap \mathbf{U}^d(\mathfrak{gl}(V))$ such that $D_\lambda = \breve{\rho}(1 \otimes z_\lambda)$. Then for a highest weight vector $v_\mu^* \otimes v_\mu \in V_\mu^* \otimes V_\mu \cong W_\mu$, we have

$$c_\lambda(\mu) v_\mu^* \otimes v_\mu = D_\lambda v_\mu^* \otimes v_\mu = v_\mu^* \otimes \breve{\rho}(z_\lambda) v_\mu = \mu(\mathrm{HC}^+(z_\lambda)) v_\mu^* \otimes v_\mu,$$

where $\mathrm{HC}^+ : \mathbf{U}(\mathfrak{gl}(V)) \to \mathbf{U}(\mathfrak{h})$ is the Harish-Chandra projection corresponding to the decomposition $\mathbf{U}(\mathfrak{gl}(V)) = (\mathbf{U}(\mathfrak{gl}(V))\mathfrak{n}^+ + \mathfrak{n}^-\mathbf{U}(\mathfrak{gl}(V))) \oplus \mathbf{U}(\mathfrak{h})$. From the description of the image of the Harish-Chandra projection (see for example [22], [24], [11], [7], or [5, Sec. 2.2.3]), we obtain explicit generators for $\mathrm{HC}^+(\mathbf{Z}(\mathfrak{gl}(V))$, similar to the G_d defined in (59). Furthermore, by the canonical isomorphism $\mathbf{U}(\mathfrak{h}) \cong \mathcal{S}(\mathfrak{h}) \cong \mathscr{P}(\mathfrak{h}^*)$, we can consider $\mathrm{HC}^+(\mathbf{Z}(\mathfrak{gl}(V))$ as a subalgebra of $\mathscr{P}(\mathfrak{h}^*)$, which we henceforth denote by $\mathbf{I}(\mathfrak{h}^*)$. A direct calculation using the explicit generators of $\mathbf{I}(\mathfrak{h}^*)$ proves that $\mathscr{F}(\mathbf{I}(\mathfrak{h}^*)) \subseteq \Lambda_{m,n,1}^{\natural}$, where $\Lambda_{m,n,1}^{\natural}$ consists of polynomials $f(\mathsf{x}_1, \ldots, \mathsf{x}_m, \mathsf{y}_1, \ldots, \mathsf{y}_n)$, which are separately symmetric in $\mathsf{x} := (\mathsf{x}_1, \ldots, \mathsf{x}_m)$ and in $\mathsf{y} := (\mathsf{y}_1, \ldots, \mathsf{y}_n)$, and which satisfy the relation

$$f(\mathsf{x} + \tfrac{1}{2}\mathsf{e}_{k,m}, \mathsf{y} - \tfrac{1}{2}\mathsf{e}_{l,n}) = f(\mathsf{x} - \tfrac{1}{2}\mathsf{e}_{k,m}, \mathsf{y} + \tfrac{1}{2}\mathsf{e}_{l,n})$$

on every hyperplane $\mathsf{x}_k + \mathsf{y}_l = 0$, where $1 \le k \le m$ and $1 \le l \le n$. Now let $SP_\flat^*$ be the basis of $\Lambda_{m,n,1}^{\natural}$ introduced in [25, Sec. 6] for $\theta = 1$. An argument similar to the proof of Theorem 6.5 yields the following statement.

Theorem B.3. *Let $\lambda \in \breve{\mathrm{E}}_{m,n,d}$, and let $\flat \in \mathrm{H}_{m,n,d}$ be chosen such that $\lambda = \mathbf{\Gamma}(\flat)$, then*

$$\mathscr{F}(c_\lambda) = \frac{d!}{\breve{H}(\flat)} SP^*_\flat.$$

References

[1] Alldridge, A.; Schmittner, S., *Spherical representations of Lie supergroups.* J. Funct. Anal. 268 (2015), no. 6, 1403–1453.

[2] Berele, A.; Regev, A., *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras.* Adv. Math. 64 (1987), no. 2, 118–175.

[3] Brini, A.; Huang, R. Q.; Teolis, A. G. B.,*The umbral symbolic method for supersymmetric tensors.* Adv. Math. 96 (1992), 123–193.

[4] Cheng, S.-J.; Wang, W., *Howe duality for Lie superalgebras.* Compositio Math. 128 (2001), 55–94.

[5] Cheng, S.-J.; Wang, W., *Dualities and representations of Lie superalgebras*, Graduate Studies in Mathematics, 144. American Mathematical Society, Providence, RI, 2012. xviii+302 pp.

[6] Goodman, R.; Wallach, N. R., *Symmetry, representations, and invariants.* Graduate Texts in Mathematics, 255. Springer, Dordrecht, 2009. xx+716 pp.

[7] Gorelik, M., *The Kac construction of the centre of U(g) for Lie superalgebras.* J. Nonlinear Math. Phys. 11 (2004), no. 3, 325–349.

[8] Howe, R., *Remarks on classical invariant theory.* Trans. Amer. Math. Soc. 313 (1989), no. 2, 539–570.

[9] Howe, R.; Umeda, T., *The Capelli identity, the double commutant theorem, and multiplicity-free actions.* Math. Ann. 290 (1991), no. 3, 565–619.

[10] Jacobson, N., *Basic algebra II.* Second edition. W. H. Freeman and Company, New York, 1989. xviii+686 pp.

[11] Kac, V. G. *Laplace operators of infinite-dimensional Lie algebras and theta functions.* Proc. Nat. Acad. Sci. U.S.A. 81 (1984), no. 2, Phys. Sci., 645–647.

[12] Kostant, B.; Sahi, S., *Jordan algebras and Capelli identities.* Inv. Math. 112 (1993), no. 3, 657–664.

[13] Knop, F., *Some remarks on multiplicity free spaces.* Representation theories and algebraic geometry (Montreal, PQ, 1997), 301–317, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Kluwer Acad. Publ., Dordrecht, 1998.

[14] Knop, F.; Sahi, S., *Difference equations and symmetric polynomials defined by their zeros.* Internat. Math. Res. Notices (1996), no. 10, 473–486.

[15] Kostant, B.; Sahi, S., *The Capelli identity, tube domains, and the generalized Laplace transform.* Adv. Math. 87 (1991), no. 1, 71–92.

[16] Molev, A., *Factorial supersymmetric Schur functions and super Capelli identities.* Kirillov's seminar on representation theory, 109–137, Amer. Math. Soc. Transl. Ser. 2, 181, Amer. Math. Soc., Providence, RI, 1998.

[17] Musson, I., *Lie superalgebras and enveloping algebras.* Graduate Studies in Mathematics, 131. American Mathematical Society, Providence, RI, 2012. xx+488 pp.

[18] Okounkov, A.; Olshanski, G., *Shifted Jack polynomials, binomial formula, and applications.* Math. Res. Lett. 4 (1997), no. 1, 69–78.

[19] Olshanskii, G. I.; Prati, M. C., *Extremal weights of finite-dimensional representations of the Lie superalgebra* $\mathfrak{gl}_{n|m}$. Nuovo Cimento A (11) 85 (1985), no. 1, 1–18.

[20] Sahi, S., *The spectrum of certain invariant differential operators associated to a Hermitian symmetric space.* Lie theory and geometry, 569–576, Progr. Math., 123, Birkhuser Boston, Boston, MA, 1994.

[21] Scheunert, M., *Eigenvalues of the Casimir operators for the general linear, the special linear, and the orthosymplectic Lie superalgebras.* J. Math. Phys. 24 (1983), no. 11, 2681–2688.

[22] Sergeev, A. N., *Invariant polynomial functions on Lie superalgebras.* C. R. Acad. Bulgare Sci. (1982), no. 5, 573–576.

[23] Sergeev, A. N., *Tensor algebra of the identity representation as a module over the Lie superalgebras* $GL(n,m)$ *and* $Q(n)$. (Russian) Mat. Sb. (N.S.) 123(165) (1984), no. 3, 422–430.

[24] Sergeev, A. N., *The invariant polynomials on simple Lie superalgebras.* Represent. Theory 3 (1999), 250–280.

[25] Sergeev, A. N.; Veselov, A. P., *Generalised discriminants, deformed Calogero–Moser–Sutherland operators and super-Jack polynomials.* Adv. Math. 192 (2005) 341–375.

[26] Weyl, H., *The Classical Groups. Their Invariants and Representations.* Princeton University Press, Princeton, N.J., 1939. xii+302 pp.

[27] Želobenko, D. P., *Compact Lie groups and their representations.* Translated from the Russian by Israel Program for Scientific Translations. Translations of Mathematical Monographs, Vol. 40. American Mathematical Society, Providence, R.I., 1973. viii+448 pp.